Supersymmetric Quantum Mechanics and Two-Dimensional Systems

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We introduce a non usual realization of the supersyrnmetric algebra that enables us to treat two-dirnensional systems in a simple way. We study the Hartmann potential as an example.

I. Introduction

Aplications of supersymmetric quantum mechanics (SQM) have been extensively explored in one[1] and more[2,3] dimensional systems. Particularly, it has been possible to determine new classes of exactly potentials by using the superalgebra[4,5,6].

Here, we intend to introduce a simple alternative realization of the superalgebra which allows us to treat quantum systems in two dimensions. This realization is applicable to Hamiltonians that can be written in two separated equations (sec. 2). We take the Hartmann potential[7] that has been used to study the benzene molecule, as an example of this realization and we find its supersymmetric version (sec. 3). From Hartmann supersymmetric Hamiltonian we construct new potentials whose spectra and eigenfunctions are related with the original Hartmann potential (sec. 4).

II. SQM with N=2

In SQM (N=2) there are two charge operators, Q and Q^+, that obey the following anticommutation relations

$$\{Q,Q\} = \{Q^+,Q^+\} = 0 \quad \text{and} \quad \{Q,Q^+\} = H_{ss} \tag{1}$$

The usual simple realization of this algebra is given by 2 x 2 matrices[4,8]. However, we can find other realization; namely,

$$Q = d_|^- \times \sigma_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^- & 0 & 0 & 0 \\ 0 & b^- & 0 & 0 \end{pmatrix}$$

$$d_|^- = \begin{pmatrix} a^- & 0 \\ 0 & b^- \end{pmatrix} \;;\; \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

$$Q^+ = d_|^+ \times \sigma_+ = \begin{pmatrix} 0 & 0 & a^+ & 0 \\ 0 & 0 & 0 & b^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;;$$

$$d_|^+ = \begin{pmatrix} a^+ & 0 \\ 0 & b^+ \end{pmatrix} \;;\; \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

Where a^- and b^- are bosonic and a^+ and b^+ are their hermitian conjugate. Thus, we get

$$H_{ss} = \begin{pmatrix} a^+a^- & 0 & 0 & 0 \\ 0 & b^+b^- & 0 & 0 \\ 0 & 0 & a^-a^+ & 0 \\ 0 & 0 & 0 & b^-b^+ \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{4}$$

In terms of the eigenfunctions, we have

$$Q\begin{pmatrix} \chi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d_|^- \chi \end{pmatrix}$$
or
$$Q\begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a^- \chi_1 \\ b^- \chi_2 \end{pmatrix} \tag{5}$$

$$Q^+\begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} = \begin{pmatrix} d_|^+ \bar{\chi} \\ 0 \end{pmatrix}$$
or
$$Q^+\begin{pmatrix} 0 \\ 0 \\ \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix} = \begin{pmatrix} a^+ \bar{\chi}_1 \\ b^+ \bar{\chi}_2 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

i.e., the operators *A* and Q^+ induce transformations between the **bosonic** sector (χ) and *fermionic* sector ($\bar{\chi}$).

We also note that

$$[H_{ss}, Q] = [H_{ss}, Q^+] = 0 \quad (7)$$

In equation (4) we can interpret H+ $= d_l^+ d_l^-$ as a supersymmetric partner of H– = i. These two Hamiltonians have the same spectra, except for the ground state, as usual in SQM.

We observe that to obtain H+ in this way we must have a Hamiltonian that can be written in terms of two separated equations.

III. Harmann Potential

The Hartmann potential[9,10] has the form

$$V(r,\theta) = \gamma\sigma^2 \left(\frac{2a_0}{r} - \frac{\gamma a_0^2}{r^2 sen^2\theta}\right)\epsilon_0 \quad (8)$$

with $a_0 = \hbar/\mu e^2$ (Bohr radius), $\epsilon_0 = -\mu e^2/\hbar^2$ (ground state H-atom energy). The parameters γ and σ are positive constants whose values usually lie in the range 1 to 10, in molecular applications. We will write the Schrodinger equation in terms of the "squared" parabolic coordinates[9].

$$\chi = \xi\eta\cos\varphi, \quad y = \xi\eta\cos\varphi, \quad r = 1/2(\eta^2 - \xi^2) \quad (9)$$

This leads us to separated equations

$$\frac{d^2\chi_1}{d\xi^2} \; \frac{M^2 - 1/4}{\xi^2}\chi_1 + \frac{2\mu E}{\hbar^2}\xi^2\chi_1 - \frac{2\mu}{\hbar}\alpha_1\chi_1 = 0 \quad (10)$$

$$\frac{d^2\chi_2}{d\eta^2} - \frac{M^2 - 1/4}{\eta^2}\chi_2 + \frac{2\mu E}{\hbar^2}\chi^2 - \frac{2\mu}{\hbar}\alpha_2\chi_2 = 0 \quad (11)$$

where the original eigenfunction was taken as $\psi(\xi,\eta,\varphi) = (\xi\eta)^{-1/2}\ \chi_1(\eta)\ \chi_2(\eta)e^{im\varphi}$ and $M^2 = m^2 + \gamma^2\sigma^2$. The parameters α_1 and α_2 obey the relation (constraint)

$$\alpha_1 + \alpha_2 = 4\gamma^2\sigma^2\epsilon_0 a_0 \quad (12)$$

We can rewrite equations (10) and (11) as

$$\left\{-\frac{d^2}{dx_1^2} + \frac{M^2 - 1/4}{x_1^2} + \frac{2\mu x_1^2}{\hbar^2}\right\} = \chi_1(x_1) = -\frac{2\mu\alpha_1}{\hbar\sqrt{|E|}}\chi_1(x_1) \quad (13)$$

$$\left\{-\frac{d^2}{dx_2^2} + \frac{M^2 - 1/4}{x_2^2} + \frac{2\mu x_2^2}{\hbar^2}\right\} = \chi_2(x_2) = -\frac{2\mu\alpha_2}{\hbar\sqrt{|E|}}\chi_2(x_2) \quad (14)$$

Where $x_1^2 = \sqrt{|E|}\xi^2$ and $x_2^2 = \sqrt{|E|}\eta^2$. From the two equations above, we can define

$$H_+ = \begin{pmatrix} a^+a^- & 0 \\ 0 & a-a+ \end{pmatrix} \equiv \begin{pmatrix} -\frac{d^2}{dx_1^2} + \frac{M^2-1/4}{x_1^2} + \frac{2\mu}{\hbar^2}x_1^2 - 2\sqrt{\frac{2\mu}{\hbar^2}}(|M|+1) & 0 \\ 0 & -\frac{d^2}{dx_2^2} + \frac{M^2-1/4}{x_2^2} + \frac{2\mu}{\hbar^2}x_2^2 - 2\sqrt{\frac{2\mu}{\hbar^2}}(|M|+1) \end{pmatrix} \quad (15)$$

Now it is clearly convenient to choose the following operators

$$a^\pm = \mp\frac{d}{dx_1} - \frac{|M|+1/2}{x_1} + \frac{\sqrt{2\mu}}{h}x_1 \quad (16)$$

$$b^\pm = \mp\frac{d}{dx_2} - \frac{|M|+1/2}{x_2} + \frac{\sqrt{2\mu}}{\hbar}x_2 \quad (17)$$

To understand the role wich the constants play, we use a^+a^- and b^+b^- as part of the original Hamiltonian H_0

$$H_0 = trH_+ = a^+a^- + b^+b^- \quad (18)$$

The eigenvalue (E_0) of H_0 is obtained from the equations (13), (14) and (18) as

$$E_0 = -\frac{2\mu}{\hbar\sqrt{|E|}}\alpha_1 - \frac{2\mu}{\hbar\sqrt{|E|}}\alpha_2 - 4(|M|+1)\frac{\sqrt{2\mu}}{h} \quad (19)$$

For the ground state[8] ($n_1 = n_2 = 0$), we have:

$$\frac{\sqrt{2\mu|E|}}{h} = -\frac{\mu}{2\hbar}\frac{\alpha_1 + \alpha_2}{|M|+1}$$

and we obtain $E_0 = 0$. We note that the constants in the components of H_0 yield a zero-energy ground state.

With the operators (16) and (17), we can define the supersymmetric partner of H+

$$H_- = \begin{pmatrix} a^-a^+ & 0 \\ 0 & b^-b^+ \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx_1^2} + \frac{M^2+2|M|+3/4}{x_1^2} + \frac{2\mu}{\hbar^2}x_1^2 - 2|M|\frac{\sqrt{2\mu}}{\hbar} & 0 \\ 0 & -\frac{d^2}{dx_2^2} + \frac{M^2+2|M|+3/4}{x_2^2} + \frac{2\mu}{\hbar^2}x_2^2 - 2|M|\frac{\sqrt{2\mu}}{\hbar} \end{pmatrix} \quad (20)$$

Where, as we have seen in Sec. 2. the eigenfunction of H_- is

$$\begin{pmatrix} \bar\chi_1 \\ \bar\chi_2 \end{pmatrix} = \begin{pmatrix} a^- & 0 \\ 0 & b^- \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = d_l^- \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad (21)$$

and the eigenvalues are the same as those of H+, except for ground state.

IV. Generalization of the Hartmann potential

As it has been done for one-dimensional cases[4,5,6], we can find new potentials from Hartmann potential.

We define new operators

$$\mathcal{D} = \begin{pmatrix} A^- & 0 \\ 0 & B^- \end{pmatrix} \quad \text{and} \quad \mathcal{D}^+ = \begin{pmatrix} A^- & 0 \\ 0 & B^- \end{pmatrix} \quad (22)$$

where

$$A^* = \mp\frac{d}{dx_1} + f(x_1) \quad \text{and} \quad B^\pm = \mp\frac{d}{dx_2} + f(x_2) \quad (23)$$

and we impose that

$$H_- = \mathcal{D}\mathcal{D}^+ = \begin{pmatrix} A^-A^+ & 0 \\ 0 & B^-B^+ \end{pmatrix} \quad (24)$$

Using equation (20), we determine the general form of the functions $f(\chi_1)$ and $f(\chi_2)$, namely,

$$f(x_1) = -\frac{|M|+1/2}{x_1} + \frac{\sqrt{2\mu}}{\hbar}x_1 + \frac{x_1^{2|M|+1}\exp\left\{-4\frac{\sqrt{2\mu}}{\hbar}x_1^2\right\}}{\Gamma_1 + \int_0^{x_1} d\bar{x}_1 \bar{x}_1^{2|M|+1}\exp\left\{-4\frac{\sqrt{2\mu}}{\hbar}\bar{x}_2^2\right\}} \equiv -\frac{|M|+1/2}{x_1} + \frac{\sqrt{2\mu}}{\hbar}x_1 + \phi(x_1) \quad (25)$$

$$f(x_2) = -\frac{|M|+1/2}{x_2} + \frac{\sqrt{2\mu}}{\hbar}x_2 + \frac{x_2^{2|M|+1}\exp\left\{-4\frac{\sqrt{2\mu}}{\hbar}x_2^2\right\}}{\Gamma_2 + \int_0^{x_2} d\bar{x}_2 \bar{x}_2^{2|M|+1}\exp\left\{-4\frac{\sqrt{2\mu}}{\hbar}\bar{x}_2^2\right\}} \equiv -\frac{|M|+1/2}{x_2} + \frac{\sqrt{2\mu}}{\hbar}x_2 + \phi(x_2) \quad (26)$$

In order to avoid singularities we choose $\Gamma_1 > 0$ and $\Gamma_2 > 0$. Thus we have the commutation relations:

$$[A^-, A^+] = 2\frac{|M|+1/2}{x_1^2} + 2\frac{\sqrt{2\mu}}{h} + 2\frac{d}{dx_1}\phi(x_1) \quad (27)$$

$$[B^-, B^+] = 2\frac{|M|+1/2}{x_2^2} + 2\frac{\sqrt{2\mu}}{\hbar} + 2\frac{d}{dx_2}\phi(x_2) \quad (28)$$

A new Hamiltonian is defined by

$$\mathcal{H} = \mathcal{D}^+\mathcal{D} = \begin{pmatrix} A^+A^- & 0 \\ 0 & B^+B^- \end{pmatrix} = \begin{pmatrix} A^-A^+ - [A^-, A^+] & 0 \\ 0 & B^-B^-[B^-, B^+] \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{d^2}{dx_1^2} + \frac{M^2+1/4}{x_1^2} + \frac{2\mu}{\hbar}x_1^2 - 2(|M|+1)\frac{\sqrt{2\mu}}{\hbar} - 2\frac{d}{dx_1}\phi(x_1) & 0 \\ 0 & -\frac{d^2}{dx_2^2} + \frac{M^2+1/4}{x_2^2} + \frac{2\mu}{\hbar}x_2^2 - 2(|M|+1)\frac{\sqrt{2\mu}}{\hbar} - 2\frac{d}{dx_2}\phi(x_2) \end{pmatrix} \quad (29)$$

and the eigenfunctions of H is written in terms of H− as

$$\mathcal{D}^+ H_- \bar{\chi} = \mathcal{D}^+ \epsilon_n \bar{\chi} \rightarrow \mathcal{H}\mathcal{D}^+\bar{\chi} = \epsilon_n \mathcal{D}^+\bar{\chi} \quad (30)$$

i.e., $\mathcal{H}$ has the same eigenfunction $\chi = \mathcal{D}^+\bar{\chi} = \mathcal{D}^+ d_|^- \chi$ and spectrum as H−, except for the ground state, which is obtained from

$$\mathcal{D}\tilde{\mathcal{X}} = 0 \Rightarrow$$

$$\tilde{\mathcal{X}} = \begin{pmatrix} x_1^{|M|+1/2}\exp\left\{-\frac{1}{2}\frac{\sqrt{2\mu}}{\hbar}\right\}\exp\left\{-\int_0^{x_1}\phi(\bar{x})d\bar{x}\right\} \\ x_2^{|M|+1/2}\exp\left\{-\frac{1}{2}\frac{\sqrt{2\mu}}{\hbar}\right\}\exp\left\{-\int_0^{x_1}\phi(\bar{x})d\bar{x}\right\} \end{pmatrix} \quad (31)$$

We can define a complete Hamiltonian

$$\mathcal{H}_c = tr(\mathcal{H}) = A^+A^- + B^+B^- \quad (32)$$

where the constraint (12) still holds here. This Hamiltonian has the same spectrum as H_0 (18) and we can construct its eigenfunctions from the knowledge of each component of χ.

V. Conclusions

We presented a realization of the superalgebra that permits to study two-dimensional systems in a simple way. The generalization to n-dimensional systems can be made in the same way by enlarging the matrices $d_|^-$ and $d_|^+$ in order to include the new equations that are written in terms of the new variable. However, we note that this approach is only applicable if the original Hamiltonian can be written in terms of separated equations, as we have done for the Hartmann potential.

In our treatment of the Hartmann potential, we have the extra constraint (12) that relates the two components of H+ (15). This is not the case when we study an uncoupled system, as for example, two independent harmonic oscillators in a plane.

We constructed the supersymmetric version of the Hartmann potential, generalized it and found out new potentials whose solutions were obtained in terms of the solution of the original systems. These new potentials have the same spectra of the original potential but different eigenfunctions. Although they are not associated to any know physical system, they can be used to study systems where the form of the eigenfunctions is important. In ref. [11] we have explained this possibility for the generalized one-dimensional harmonic-oscillator.

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