Low-energy theorems in non-Abelian Compton scattering on targets of arbitrary spin

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Received January 7, 1991, in final form June 30, 1991

Abstract The low-energy theorems in non-Abelian Compton scattering previously obtained for spin-1/2 targets are extended to the case of targets of arbitrary spin.

1. Introduction

In a previous paper we have obtained four new isospin-antisymmetric low-energy theorems in nucleon non-Abelian Compton scattering by using a second lemma for the Compton amplitude. In this note we shall use it to discuss the scattering on targets of arbitrary spin. Lin has considered the case of normal Abelian Compton scattering. We shall then concentrate only in the isospin-antisymmetric part of the amplitude and show that seven new low-energy theorems can be obtained to second order in the frequency of the incoming photon. The basic equation is Eq.(4) of Ref.1, which reads

\[ k'_{i}(E^{\alpha\beta} - \omega') \Gamma^{\alpha\beta}_{mn} = -k'_{m} U^{\alpha\beta}_{mn} + \omega' U^{\alpha\beta}_{on} + i(2\pi)^{3} \left( \frac{E E'}{M^{2}} \right)^{1/2} \epsilon^{\alpha\beta\gamma} < \vec{p}' |J_{\gamma_{n}}| \vec{p'} >. \]

In this equation \( E^{\alpha\beta}_{mn} \) is the space-space component of the excited part of the amplitude tensor and \( U^{\alpha\beta}_{\mu\nu} \) is the unexcited part. \( \vec{k}'(\omega') \) is the momentum (energy) of the outgoing photon, \( \vec{k} (\omega) \) will designate the corresponding quantities for the incoming photon and \( \vec{p}, E(\vec{p}', E') \) are the initial (final) nucleon momentum and
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Energy respectively. \( J'_\nu \) is the electromagnetic current and \( \Gamma^\alpha_\beta_{mn} \) is a quantity which is odd under crossing in the Breit frame, where

\[
\vec{p}' = -\vec{p} = -\frac{\vec{k}' - \vec{k}}{2}, \quad E' = E, \quad \omega' = \omega.
\] (2)

2. Expansion of \( B^{\alpha\beta}_{mn} \) in the Breit frame

The Breit frame is the frame where T-reversal invariance achieves its simplest form as used by Pais to write down the minimal basics \( B^{(i)}_{mn} \) in which \( B^{\alpha\beta}_{mn} \) is to be expanded. To order \( \omega^2 \) we have, for the isospin-antisymmetric part,

\[
B^{[\alpha\beta]}_{mn} = [I^\alpha, I^\beta] \sum_i a_i B^{(i)}_{mn}
\]

\[
= [I^\alpha, I^\beta](\omega a_{1,1} \delta_{mn} + a_2(0) + a_{2,1} \bar{k} \vec{k}' + a_{2,2} \omega^2 \epsilon_{mnj} S_j \\
+ \omega a_{3,1} (S_m S_n + S_n S_m - \frac{2}{3} \vec{S}^2 \delta_{mn}) \\
+ a_4(0)[\delta_{mn} \vec{S}.(\vec{k}' \times \vec{k}) - \bar{k} \vec{k}' \epsilon_{mnj} S_j] \\
+ a_5(0)[k_m (\vec{S} \times \vec{k})_n - k'_n (\vec{S} \times \vec{k}')_m] \\
+ a_6(0)[k'_m (\vec{S} \times \vec{k}')_n - k_n (\vec{S} \times \vec{k})_m - 2 \omega^2 \epsilon_{mnj} S_j] \\
+ a_7(0)[k_m (\vec{S} \times \vec{k})_n - k'_n (\vec{S} \times \vec{k}')_m - 2 \vec{k} \vec{k}' \epsilon_{mnj} S_j] \\
+ a_8(0)[k'_m (\vec{S} \times \vec{k}')_n - k_n (\vec{S} \times \vec{k})_m] \\
+ a_9(0)[<S_m, S_n, \vec{S}.(\vec{k}' \times \vec{k})> + <S_n, S_m, \vec{S}.(\vec{k}' \times \vec{k})>- \\
- \frac{2}{5}(3 \vec{S}^2 - 1) \vec{S}.(\vec{k}' \times \vec{k}) \delta_{mn}] \\
+ a_{10}(0) \epsilon_{mnj}[<\vec{S} \vec{k}', \vec{S} \vec{k}, S_j > + <\vec{S} \vec{k}, \vec{S} \vec{k}, S_j >]
\]
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\[ + a_{11}(0) \epsilon_{mnj} [\langle \vec{S}, \vec{k}, \vec{S}, \vec{k}, S_j \rangle + \langle \vec{S}, \vec{k}, \vec{S}, \vec{k}, S_j \rangle - \]

\[ - \frac{2}{5} (3 \vec{S}^2 - 1) \vec{S}.(\vec{k} \times \vec{k}) \delta_{mn} ] \]

\[ + a_{12}(0)[\langle S_m, (\vec{S} \times \vec{k})_n, \vec{S}.\vec{k} \rangle - \langle S_n, (\vec{S} \times \vec{k})_m, \vec{S}.\vec{k} \rangle ] \]

\[ + a_{13}(0)[\langle S_m, \vec{S}.\vec{k}, (\vec{S} \times \vec{k})_n \rangle - \langle S_n, \vec{S}.\vec{k}, (\vec{S} \times \vec{k})_m \rangle ] \]

\[ + a_{14}(0)[\langle S_m, (S \times C)_n, \vec{S}.\vec{k} \rangle - \langle S_n, (S \times \bar{C})_m, \vec{S}.\vec{k} \rangle - \]

\[ - (3 \vec{S}^2 - 1)(B^{(7)}_{mn} + B^{(8)}_{mn}) ] \]

\[ + a_{15}(0)[\langle S_m, \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n \rangle - \langle S_n, \vec{S}.\vec{k}', (\vec{S} \times \vec{k}')_m \rangle - \]

\[ - (3 \vec{S}^2 - 1)(B^{(7)}_{mn} + B^{(8)}_{mn}) ] \]

(3)

where

\[ \langle A, B, C \rangle = ABC + CAB + BCA \]  

(4)

\[ \vec{S}^2 = S(S + 1), I^a \]  

is the ath component of isospin, and the expansion of the coefficients, to order \( w^2 \), is an accordance with the even crossing symmetry property of \( E_{mn}^{ab} \). In Ref.4 the expansion in Eq.(12) has been done in the Lab frame and the last four basis elements \( B^{12}_{mn} - B^{15}_{mn} \) are actually missing in that equation. However, the conclusion of that paper continues to be valid because they are based on a relation for \( k_m^ik_nE_{mn} \) which does not contain \( B^{12}_{mn} - B^{15}_{mn} \), since these basis elements are orthogonal to \( k_m^ik_n \). One can show that the other possible parity-and time-reversal invariant basis elements obtained by interchanging \( \vec{k} \) and \( \vec{k}' \) in \( B^{(12)}_{mn} \) to \( B^{(15)}_{mn} \) are not linearly independent, as we discuss in the Appendix. For convenience we have used parity-and time-reversal-invariant amplitudes basis elements \( P^{i}_{mn} \) for \( i = 5-8 \) of which the corresponding basis elements \( E^{(i)}_x \) of Pais \( ^3 \) there numbered \( i = 8-11 \), are linear combinations, as given by Eqs.(A.7) to (A.10) of the Appendix.
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3. The Low-Energy Theorems

The usual procedure to obtain low-energy theorems uses a relation involving \( k'_m k_n E^\alpha_{mn} \) and then it cannot give information on the partial amplitudes for which the corresponding basis elements are orthogonal to \( k'_m k_n \) that is, \( k'_m k_n E^{(f)}_{mn} = 0 \).

This is the case for \( i = 4-8 \) and \( i = 12-15 \) in Eq.(3), which will now be accessible to us. We shall then concentrate only on this part of the amplitude.

As \( \Gamma^\alpha_{mn} \) is a three-tensor as is \( E^\alpha_{mn} \), it can be expanded in the same minimal basis. To get information about \( E^\alpha_{mn} \) to order \( w^2 \) we see from Eq.(1) that we need to consider \( \Gamma^\alpha_{mn} \) to order \( w \). As \( \Gamma^\alpha_{mn} \) is odd under crossing we have, to order \( w \),

\[
\Gamma^\alpha_{mn} = [I^\alpha, I^\beta]\left\{b_1(0)\delta_{mn} + \omega b_{21} \epsilon_{mnp} S_j + b_3(0)\left(S_m S_n + S_n S_m - \frac{2}{3} S^2 \delta_{mn}\right)\right\}
\]

(5)

Using Eqs. (3) and (5) and the identity (A.15) of the Appendix the isospin-antisymmetric part of the left-hand side of Eq.(3), which contains the terms that are of interest to us (\( i = 4-8 \) and \( i = 12-15 \)), become

\[
k'_m (E^\alpha_{mn} - \omega \Gamma^\alpha_{mn}) \rightarrow [I^\alpha, I^\beta]\left\{k'_n \bar{S}.(\vec{k} \times \vec{k})|a_4(0) + a_7(0) + \right.
\]

\[
\frac{2}{5} (3 \bar{S}^2 - 1)(a_9(0) + a_{11}(0)) + (\bar{S} \times \vec{k})_n \bar{k} \cdot \vec{k} |a_6(0) + \right.
\]

\[
+ k_n \bar{S}.(\vec{k} \times \vec{k}) |a_6(0) +
\]

\[
[< \bar{S} \cdot \vec{k}, S_n, \bar{S}.(\vec{k} \times \vec{k}) > + < \bar{S} \cdot \vec{k}, \bar{S} \cdot (\vec{k} \times \vec{k}) >] |a_9(0) - \frac{1}{2} (a_{14}(0) +
\]

\[
a_{15}(0)) + [< \bar{S} \cdot \vec{k}, \bar{S} \cdot \vec{k}, (\bar{S} \times \vec{k})_n + < \bar{S} \cdot \vec{k}, \bar{S} \cdot \vec{k}, ((\bar{S} \times \vec{k})_n > |a_{10}(0)
\]

\[
+ [< \bar{S} \cdot \vec{k}, \bar{S} \cdot \vec{k}, (\bar{S} \times \vec{k})_n > + < \bar{S} \cdot \vec{k}, \bar{S} \cdot \vec{k}, ((\bar{S} \times \vec{k})_n > |a_{11}(0) + \frac{1}{2} (a_{14}(0) +
\]

\[
a_{15}(0)) + [< \bar{S} \cdot \vec{k}, (\bar{S} \times \vec{k})_n, \bar{S} \cdot \vec{k} > a_{12}(0) + < \bar{S} \cdot \vec{k}, \bar{S} \cdot \vec{k}, (\bar{S} \times \vec{k})_n > a_{13}(0)
\]

(6)

All the coefficients that are present in this equation will be determined by the right-hand side of Eq.(1), giving rise to nine low-energy theorems. The coefficients \( a_{10}(0) \) and the sum \( a_9(0) + a_{11}(0) \) have been determined before\(^4\) by considering a relation for \( k'_m k_n E^{(f)}_{mn} \). The rest constitutes the new seven low-energy theorems and we shall concentrate only on them.

To establish the expression of the new lowenergy theorems we have to calculate the right-hand side of Eq.(1) in the Breit frame. For that purpose we need the
expression of the current matrix element for arbitrary spin. This has been determined before to second order in terms of the related form factors and multipole moments, which are the charge and magnetic moment mean square radii $<r^2>V$ and $<R^2>V$, respectively, the isovector magnetic dipole moment $\mu^V$ (in units of $1/2M$), the electric quadrupole moment $Q^V$ (in units of $1/M^2$) and the magnetic octupole moment $\Omega^V$ (in units of $\sqrt{6}/2M^3$), defined by Eqs. (17a-e) of Ref.4. Calculating the right-hand side of Eq.(1) and comparing with Eq.(6) we obtain as new isospin-antisymmetric low-energy theorems,

$$a_4(0) + a_7(0) = \frac{i}{8m^3} \left( \frac{3\mu^V}{2S} - 2 \right)$$

$$ + \frac{i}{4M^3S(2S-1)} \left[ \frac{(2S^2-1)\mu^VQ^V}{2S} - \frac{7(3\bar{S}^2-1)Q^V}{15(S-1)} \right], \quad (7)$$

$$a_5(0) = \frac{1}{i} \left[ \frac{\mu^V}{16M^3S(2S-1)} (\frac{\mu^V}{S} - 1) \right]$$

$$- \frac{<R^2>}{5S} + \frac{(3S^2-1)\Omega^V}{30M^3S(S-1)(2S-1)} \right], \quad (8)$$

$$a_6(0) = \frac{\mu^V}{16iM^3S} \left( \frac{\mu^V}{S} - 1 \right), \quad (9)$$

$$a_8(0) = \frac{<R^2>}{10iS} + \frac{i\mu^V}{2MS} \left( \frac{<r^2>V}{6} + \frac{1}{8M^2} \right)$$

$$+ \frac{i}{3M^3S(2S-1)} \left[ \frac{(2\bar{S}^2-1)\mu^VQ^V}{4S} - \frac{(3\bar{S}^2-1)\Omega^V}{5(S-1)} \right], \quad (10)$$

$$a_9(0) - \frac{1}{2} (a_{14}(0) + a_{15}(0)) = \frac{i}{24M^3S(2S-1)} \left( \frac{\Omega^V}{S-1} - \frac{\mu^VQ^V}{S} \right), \quad (11)$$

and

$$a_{12}(0) = a_{13}(0) = \frac{i\Omega^V}{12M^3S(S-1)(2S-1)}, \quad (12)$$

where use has been made of the identities (A.17) and (A.18) of the Appendix.

For $S = 1/2$ it will be present only the results without the terms proportional to $Q^V$ and $\Omega^V$. It is easy to see that they agree with those already derived for spin-1/2 targets by expressing $<r^2>V$ and $<R^2>V$ for $S = 1/2$ in terms of $\mu^V$ and the nucleon form factors.
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Acknowledgement

The author wishes to thank the referee for useful suggestions that helped the presentation of the paper.

Appendix

By expanding $\tilde{S} \cdot [(\tilde{S} \times \tilde{k}) \times (\tilde{p} \times \tilde{q})]$ and $\tilde{S} \cdot [(\tilde{q} \times (\tilde{a} \times \tilde{b}))$ in two different ways one easily establishes the identities

$$\tilde{S} \cdot k \tilde{S} \cdot (\tilde{p} \times \tilde{q}) - \tilde{S}^2 \tilde{k} \cdot (\tilde{p} \times \tilde{q}) = \tilde{S} \cdot p \tilde{S} \cdot (\tilde{k} \times \tilde{q}) - \tilde{S} \cdot q \tilde{S} \cdot (\tilde{k} \times \tilde{p}) \quad (A.1)$$

and

$$\tilde{S} \cdot q \tilde{k} \cdot (\tilde{a} \times \tilde{b}) - \tilde{S} \cdot (\tilde{a} \times \tilde{b}) \tilde{k} \cdot \tilde{q} = \tilde{S} \cdot (\tilde{k} \times \tilde{a}) \tilde{q} \cdot \tilde{b} - \tilde{S} \cdot (\tilde{k} \times \tilde{b}) \tilde{q} \cdot \tilde{a} \quad (A.2)$$

Using

$$S_m S_n - S_n S_m = i \epsilon_{m nj} S_j \quad (A.3)$$

we also have, from (A.1),

$$\tilde{S} \cdot (\tilde{p} \times \tilde{q}) \tilde{S} \cdot k - \tilde{S}^2 \tilde{k} \cdot (\tilde{p} \times \tilde{q}) = \tilde{S} \cdot (\tilde{k} \times \tilde{q}) \tilde{S} \cdot \tilde{p} - \tilde{S} \cdot (\tilde{k} \times \tilde{p}) \tilde{S} \cdot \tilde{q} \quad (A.4)$$

a. Using these identities one easily shows that the other possible basis elements obtained by interchanging $\tilde{k}$ and $\tilde{k}'$ in $B^{(i)}_{mn}$ for $i = 12 - 25$ are not linearly independent basis elements. Calling them $A^{(i)}_{mn}$ we have for instance for $i = 12$ and with a convenient contraction with arbitrary vectors $a$ and $b$,

$$a_m b_n A^{(12)}_{mn} = <\tilde{S} \cdot a, \tilde{S} \cdot (\tilde{k}' \times \tilde{b})>, \tilde{S} \cdot \tilde{k}'> - <\tilde{S} \cdot b, \tilde{S} \cdot (\tilde{k} \times \tilde{a})>, \tilde{S} \cdot \tilde{k}'> \quad (A.5)$$

Using Eqs. (A.1), (A.3) and (A.4) the first term on the right-hand side of Eq.(A.5) can be written as

$$<\tilde{S} \cdot a, \tilde{S} \cdot (\tilde{k}' \times \tilde{b})>, \tilde{S} \cdot \tilde{k}'> = <\tilde{S} \cdot b, \tilde{S} \cdot (\tilde{k} \times \tilde{a})>, \tilde{S} \cdot \tilde{k}'> + <\tilde{S} \cdot \tilde{k}', \tilde{S} \cdot (\tilde{a} \times \tilde{b})>, \tilde{S} \cdot \tilde{k}'> - (3 \tilde{s}^2 - 1) \tilde{S} \cdot \tilde{k}' \tilde{k} \cdot (\tilde{a} \times \tilde{b})$$

A similar relation holds for the second term on the right-hand side of Eq.(A.5). Adding the two terms and making use of (A.2) one obtains

$$A^{(12)}_{mn} = B^{(10)}_{mn} - B^{(12)}_{mn} + (3 \tilde{s}^2 - 1) (B_{mn}^{(5)} + B_{mn}^{(6)}) \quad (A.6)$$
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Similar considerations applies to the other cases, i = 13 - 15.

b. By contracting $E^{(8)}_{mn}$ of Pais with $a_m b_n$ for convenience and using Eq.(A.2) it is easy to see that

$$E^{(8)}_{mn} = -B^{(6)}_{mn} - B^{(6)}_{mn}$$

(A.7)

Also

$$E^{(9)}_{mn} = -B^{(7)}_{mn} - B^{(9)}_{mn}$$

(A.8)

As mentioned before\(^1\) the plus ans minus signs in both equations for $i = 10$ and $i = 11$ of Pais should be interchanged otherwise they would not be independent basis elements. Correcting these misprints they are related to our elements by the equations

$$E^{(10)}_{mn} = B^{(5)}_{mn} - B^{(6)}_{mn} - (\omega^2 + \omega^4)B^{(2)}_{mn}$$

(A.9)

and

$$E^{(11)}_{mn} = B^{(7)}_{mn} - B^{(8)}_{mn} + 2\bar{k}.\bar{k}B^{(2)}_{mn}$$

(A.10)

c. Callnow

$$X_n = \langle \vec{S}, \vec{k'}, \vec{S}, \vec{k}, (\vec{S} \times \vec{k'})_n \rangle + \langle \vec{S}, \vec{k}, \vec{S}, \vec{k'}, (\vec{S} \times \vec{k'})_n \rangle$$

(A.11)

and

$$Y_n = \langle S_n, \vec{S}, \vec{k'}, \vec{S}, \vec{k}, \vec{S}, (\vec{k'} \times \vec{k})_n \rangle + \langle \vec{S}, \vec{k}, S_n, \vec{S}, (\vec{k'} \times \vec{k}) \rangle$$

(A.12)

Using Eqs.(A.1) and (A.3) it follows the relation

$$X_n - Y_n = 2 \langle \vec{S}, \vec{k'}, \vec{S}, \vec{k'}, (\vec{S} \times \vec{k})_n \rangle - 2(3S^2 - 1)\vec{S}.\vec{k'}(\vec{k'} \times \vec{k})_n$$

(A.13)

From Eq.(A.2) we also have

$$\vec{S}.\vec{k'}(\vec{k'} \times \vec{k})_n = k'_m (B^{(8)}_{mn} + B^{(7)}_{mn})$$

(A.14)

Therefore,

$$\langle \vec{S}, \vec{k'}, \vec{S}, \vec{k'}, (\vec{S} \times \vec{k})_n \rangle = -(3S^2 - 1)k'_m (B^{(8)}_{mn} + B^{(7)}_{mn}) = \frac{1}{2}(X_n - Y_n)$$

(A.15)
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d. Using (A.3) we can write

\[ 2\tilde{S}.\tilde{k}'(\tilde{S} \times \tilde{k})_n\tilde{S}.\tilde{k}' = \{(\tilde{S} \times \tilde{k})_n,(\tilde{S}.\tilde{k}')^2\} + \tilde{S}.(\tilde{k}' \times \tilde{k})\tilde{k}'_n - \tilde{k}.\tilde{k}'(\tilde{S} \times \tilde{k}')_n. \] (A.16)

From here we have

\[ <\tilde{S}.\tilde{k}',\tilde{S}.\tilde{k}',(\tilde{S} \times \tilde{k})_n> = \frac{3}{2}\{(\tilde{S} \times \tilde{k})_n,(\tilde{S}.\tilde{k}')^2\} + \frac{1}{2}(\tilde{S}.(\tilde{k}' \times \tilde{k})\tilde{k}'_n - \tilde{k}.\tilde{k}'(\tilde{S} \times \tilde{k}')). \] (A.17)

Using (A.15) in (A.17) we obtain

\[ \{(\tilde{S} \times \tilde{k})_n,(\tilde{S}.\tilde{k}')^2\} = \frac{1}{3}(X_n - Y_n) + \frac{2}{3}w^2(\tilde{S} \times \tilde{k})_n \]

\[ (2\tilde{S}^2 - 1)((\tilde{S}.\tilde{k}' \times \tilde{k})\tilde{k}'_n - \tilde{k}.\tilde{k}'(\tilde{S} \times \tilde{k}')). \] (A.18)

References

4. S. Ragusa, Phys. Rev. D8, 1190 (1973). In the third line of Eq.(21) one should read \( q^2/16M^2 \) instead of \( q^2/16M^3 \).

Resumo

Os teoremas de baixa energia em espalhamento Compton não-abeliano, previamente obtidos para alvos de spin 1/2, são estendidos para o caso de alvos com spin arbitrário.