On Generalized Elliptic-Type Integrals

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Received in 23 de junho de 1987

Abstract This work is in continuation of our previous papers [Kalla and Al-Saqabi: this journal, Vol. 16, pp.145-156, 1986; Kalla: Mathematical Structures, Computational Mathematics - Math. Modelling 2, Sofia, 1984, p.216-219]. By using the differencing technique, we express generalized elliptic-type integrals in terms of confluent hypergeometric functions. The method of steepest descent is employed to obtain relations between \( K_\mu(k,m) \), \( S_\mu(k,v) \) and incomplete gamma functions. We tabulate these elliptic-type integrals by using suitable formulae. Some known results follow as particular cases of our formulae established here.

1. INTRODUCTION

In a recent paper\(^1\), the authors have studied a family of integrals

\[ K_\mu(k,m) = \int_0^\pi \frac{\cos^{2m} \theta}{(1-k^2 \cos \theta)^{\mu+\frac{1}{2}}} \, d\theta \]  

(1)

where \( 0 \leq k < 1 \), \( \Re(\mu) > -\frac{1}{2} \) and \( m \) is a non-negative integer. For \( m = 0 \) and \( \mu = j \), a positive integer

\[ K_j(k,0) = \Omega_j(k) \]  

(2)

where

\[ \Omega_j(k) = \int_0^\pi (1-k^2 \cos \theta)^{-j-\frac{1}{2}} \, d\theta \]  

(3)

(0 \leq k < 1), a family of integrals considered by Epstein and Hubbell\(^2\). Such integrals are found in the application of the Legendre polynomial expansion method\(^3\) to certain problems involving computation of the radiation
field off-axis from a uniform circular disc radiating according to an arbitrary distribution law.

Kalla\textsuperscript{4} and Kalla, Conde and Rubbell\textsuperscript{5} have defined and studied certain class of generalized elliptic-type integrals. The former are defined as

\[ S_\mu(k,\nu) = \int_0^\pi \frac{\sin^{2\nu} \theta \, d\theta}{(1-k^2 \cos \theta)^{\mu+\frac{1}{2}}}, \quad (4) \]

\[ 0 < k < 1, \Re(\mu) > -\frac{1}{2}, \Re(\nu) > -\frac{1}{2}. \]

We observe that,

\[ K_0(k,0) = \Omega_0(k) = (\sqrt{2} \rho/k)K(\rho) \]

\[ K_1(k,0) = \Omega_1(k) = (\sqrt{2} \rho/k(1-k^2))E(\rho) \]

\[ S_0(k,0) = \Omega_0(k) = (\sqrt{2} \rho/k^2)K(\rho) \]

\[ S_1(k,0) = \Omega_1(k) = (\sqrt{2} \rho/k(1-k^2))E(\rho), \]

\[ \rho^2 = 2k^2/(1+k^2) \]

where \( K(\lambda) \) and \( E(\lambda) \) are the complete elliptic integrals of the first and second kind respectively [6, p.295; 7, p.587].

In the previous paper, we obtained a series expansion of \( K_\mu(k,m) \) for small values of \( k \), and established its relationship with Gauss' hypergeometric function. Asymptotic expansions, valid in the neighbourhood of \( k^2=1 \) and some recurrence relations were given.

In this work, we continue our study of the family of integrals eq.(1) and eq.(4). As will also appear in the following sections, the families of integrals \( K_\mu(k,m) \) and \( S_\mu(k,\nu) \) are related to an interesting class of transcendental functions. By an appeal to the differentiating technique, developed for improving the convergence of series\textsuperscript{8}, we establish their relation with the confluent hypergeometric functions \textsuperscript{7,9,10,11}. The method of steepest descent is used to obtain relations with
incornplete gamma functions$^7,9,10,11$. Some particular cases are mentioned. A number of results obtained earlier by Epstein and Hubbell$^2$ follow as special cases of our formulae. We tabulate $S_\mu(k,\nu)$ by using its series expansion. We also compute $K_\mu(k, m)$ using two formulae, established in our previous paper.

It is interesting to observe that many formulae established here and in our previous paper for $K_\mu(k, m)$ ($m$; integer), can be extended to arbitrary values of $m$.

2. THE DIFFERENCING TECHNIQUE

The behaviour of series expansions can be improved by using a technique developed earlier by Epstein and French$^6$ for improving the convergence of series. The technique is to express the given integral in the following form

$$\int_a^b f(\theta) \, d\theta = \int_a^b f^*(\theta) \, d\theta + \int_a^b (f(\theta) - f^*(\theta)) \, d\theta$$

where $f^*(\theta)$ is an integrable function over the interval $a \leq \theta \leq b$, and $f(\theta)$ is a suitable approximation to $f(\theta)$.

In our case,

$$f^*(\theta) = \frac{\cos^{2m} \theta}{(1-k^2 \cos \theta)^{\mu+\frac{1}{2}}}$$

Let

$$g(\theta) = (1-k^2 \cos \theta)^{-\mu} - \frac{1}{2}$$

$$= (1-k^2 \cos \theta)^{-\lambda}, \quad \lambda = \mu + \frac{1}{2}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\lambda}{n!}\right) (k^2 \cos \theta)^n$$

$$= e^{\lambda(k^2 \cos \theta)} + \frac{\lambda}{2} (k^2 \cos \theta)^2 e^{\lambda(k^2 \cos \theta)}$$

Hence a good approximation for $f^*(\theta)$ can be
and consequently,
\[
\int_0^\pi f^*(\theta)\,d\theta = \int_0^\pi \cos^{2m}\theta e^{(\lambda + \frac{1}{2}) (k^2 \cos \theta)} e^{(\lambda + \frac{1}{2}) (k^2 \cos \theta)} d\theta \\
+ \frac{1}{2} (\mu + \frac{1}{2}) k^4 \int_0^\pi \cos^{2m+2} \theta e^{(\mu + \frac{1}{2}) (k^2 \cos \theta)} d\theta.
\]  
(13)

Using the transformation \(\cos \theta = 1 - 2w\) and a binomial expansion, we obtain after some simplification,
\[
\int_0^\pi f^*(\theta)\,d\theta = \sum_{n=0}^{2m} \binom{2m}{n} (-1)^n \int_0^1 w^n - \frac{1}{2} (1-w)^{-\frac{1}{2}} e^{(\mu + \frac{1}{2}) k^2 (1-2w)} \,dw \\
+ (\mu + \frac{1}{2}) k^4 \sum_{n=0}^{2m+2} (-1)^n \binom{2m+2}{n} 2n-1 \int_0^1 \frac{w^n}{2} (1-w)^{-\frac{1}{2}} e^{(\mu + \frac{1}{2}) k^2 (1-2w)} \,dw.
\]  
(14)

by an appeal to the integral representation for the confluent hypergeometric function\(^0,10,11\)
\[
\phi(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma-\alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\gamma-\alpha-1} \,dt
\]  
(16)

\(\text{Re}(\gamma) > \text{Re}(\alpha) > 0\).
Now, let's consider
\[
\int_0^\pi (f'(\theta) - f^{(*)}(\theta)) \, d\theta = \int_0^\pi \frac{\cos 2m_\theta \, d\theta}{(1-k^2 \cos \theta)^{\mu + \frac{1}{2}}} - \int_0^\pi \cos 2m_\theta \, (\mu + \frac{1}{2}) (k^2 \cos \theta) \, d\theta
\]
\[
- \frac{1}{2} (\mu + \frac{1}{2}) k^4 \int_0^\pi \cos 2m_\theta \, (\mu + \frac{7}{6}) (k^2 \cos \theta) \, d\theta
\]
\[
= I_1 - I_2 - I_3 \quad \text{, say} \quad (17)
\]

Further,
\[
I_1 = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^{2n} \sqrt{n} \Gamma(m+n+\frac{1}{2})}{(2n)! \Gamma(m+n+1)} \left(\frac{k^4}{2}\right)^n , \quad (18)
\]

a result given earlier by Kalla and Al-Saqabi.

\[
I_2 = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^n}{n!} \frac{k^{2n}}{\Gamma(n)} \int_0^\pi \cos 2n\theta \, d\theta . \quad (19)
\]

As
\[
\int_0^\pi \cos^\ell \theta \, d\theta = \frac{\Gamma(\frac{\ell+1}{2})}{\Gamma(\frac{\ell+2}{2})} \left(\frac{\sqrt{\pi}}{2}\right) \Gamma(\frac{\ell+1}{2})
\]

we get
\[
I_2 = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^{2n} k^{4n} \Gamma(m+n+2 \pi)}{(2n)! \left[\Gamma(m-n) \right]^2 2^{2m+2n}} . \quad (20)
\]

Similarly, we have
\[
I_3 = \frac{1}{2} (\mu + \frac{1}{2}) k^4 \sum_{n=0}^{\infty} \frac{(\mu + \frac{7}{6})^{2n} k^{4n}}{(2n)! \left[\Gamma(m-n+1) \right]^2 2^{2m+2n+2}} \quad (21)
\]

Hence
\[ \int_0^\pi (f(\theta) - f^*(\theta)) d\theta = \pi \sum_{n=0}^{\infty} \frac{(2m+2n)!}{(2n)!2^{2m+2n}[(m+n)!]} \frac{k^{2n}}{2} \]

\[ = \left[ (\mu + \frac{1}{2})^2 2n - (\mu + \frac{1}{2})^{2n} - \frac{(2m+2n+1)}{2(m+n+1)} \right] \times \frac{1}{2} (\mu + \frac{1}{2}) k^n (\mu + \frac{7}{6})^{2n} \]

\[ \equiv S(m,k,\mu) . \quad \text{(22)} \]

Thus

\[ \int_0^\pi \frac{\cos 2n \theta}{(1-k^2 \cos \theta)^{m+1}} d\theta = R(m,k,\mu) + S(m,k,\mu) \quad \text{(23)} \]

where \(R(m,k,\mu)\) and \(S(m,k,\mu)\) are given by eq. (15) and eq. (22) respectively.

**Particular Case:** For \(m=0\), eq. (15) reduces to

\[ \int_0^\pi f^*(\theta) d\theta = \int_0^\pi (\mu + \frac{1}{2}) (k^2 \cos \theta) + \frac{1}{2} (\mu + \frac{1}{2}) (k^2 \cos \theta)^2 \]

\[ = \pi \left[ I_0 \left( (\mu + \frac{1}{2}) k^2 \right) + \frac{k^4}{4} (2\mu+1) I_0 \left( (\mu + \frac{7}{6}) k^2 \right) \right] - \frac{k^2}{4} \frac{(\mu + 1)}{(\mu + \frac{7}{6})} I_1 \left( (\mu + \frac{7}{6}) k^2 \right) \quad \text{(24)} \]

where \(I_0\) and \(I_1\) are the modified Bessel functions of the first kind\(^{10,11}\).

To obtain eq. (24) from eq. (15), we have used the following results\(^{10}\)

\[ I_{\nu}(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} e^{-z} \phi \left( \nu + \frac{1}{2}, 2\nu+1, 2z \right) \quad \text{(25)} \]
Further, we observe from eq. (22), that \( m = 0, v = j \), an integer, leads to

\[
\frac{1}{\pi} \int_0^\pi (f(\theta) - f^*(\theta)) d\theta = \sum_{n=0}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2} \left\{ \frac{(2j+4n)!}{(j+2n)!} - 2^{2n}(2j+1)^{2n} \right\}
\]


\[
= \sum_{n=2}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2} \left\{ \frac{(2j+4n)!}{(j+2n)!} - 2^{2n}(2j+1)^{2n} \right\}
\]

\[
- 18 \left( \frac{2}{3} \right)^2 (2j+1)^{2n-2} \frac{2n-2}{(2n-1)(2n-2)}
\]

(27)

Eqs. (24) and (27) are in agreement with the results of Epstein and Hubbel12.

3. \( K_\mu(k,m) \) IN TERMS OF INCOMPLETE GAMMA FUNCTIONS

Let

\[
F(\theta) = (\mu + \frac{1}{2}) \log(1 - k^2 \cos \theta)
\]

(28)

hence

\[
K_\mu(k,m) = \int_0^\pi e^{-F(\theta)} \cos 2m\theta d\theta
\]

(29)

Expanding \( F(\theta) \) in Maclaurin series, we observe that all the odd derivatives of \( F(\theta) \) vanish at the origin and we can write

\[
F(\theta) = F(0) + \frac{F(2)(0)}{2!} \theta^2 + \frac{F(4)(0)}{4!} \theta^4 + \frac{F(6)(0)}{6!} \theta^6 + \ldots
\]

(30)

then
\[ K_\mu(k,m) = e^{-F(0)} \sum_{n=0}^{\infty} \frac{F^{(2)}(0)}{2^n} \theta^n \left[ 1 - \frac{F^{(4)}(0)}{4!} \theta^4 \right] \frac{F^{(6)}(0)}{6!} \theta^6 - \ldots \] \cos 2m \theta \, d\theta \quad (31) \]

\[ F(0) = (\mu + \frac{1}{2}) \log(1 - k^2) \, . \]

Let
\[ \left( \frac{\lambda}{\pi} \right)^2 = \frac{F^{(2)}(0)}{2!} \]

then
\[ K_\mu(k,m) = (1 - k^2)^{-\left(\mu + \frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{\pi}\right)^2 \theta^2}{n!} \left[ 1 - \frac{F^{(4)}(0)}{4!} \theta^4 \right] \frac{F^{(6)}(0)}{6!} \theta^6 - \ldots \] \cos 2m \theta \, d\theta \quad (32) \]

If we write \( t = \lambda \theta / \pi \) and let
\[ L_\mu(p,m)(\lambda) = \int_0^\lambda t^2 e^{-t^2} \cos 2m \left( \frac{\pi t}{\lambda} \right) \, dt \quad (33) \]

then
\[ K_\mu(k,m) = \pi(\lambda)^{-1} (1 - k^2)^{-\left(\mu + \frac{1}{2}\right)} \left[ 1 - \frac{F^{(4)}(0)}{4!} \left( \frac{\pi t}{\lambda} \right)^4 - \frac{F^{(6)}(0)}{6!} \left( \frac{\pi t}{\lambda} \right)^6 - \ldots \right] \, dt \]

\[ = \pi(\lambda)^{-1} (1 - k^2)^{-\left(\mu + \frac{1}{2}\right)} \left[ L_\mu(0,m)(\lambda) - \frac{F^{(4)}(0)}{4!} \left( \frac{\pi t}{\lambda} \right)^4 L_\mu(2,m)(\lambda) - \frac{F^{(6)}(0)}{6!} \left( \frac{\pi t}{\lambda} \right)^6 L_\mu(3,m)(\lambda) - \ldots \right] \quad (34) \]
Using the result

\[ \cos^{2m}A = 2 - 2m \left( \frac{2m}{m} \right) + 2^{1-2m} \sum_{r=1}^{m} \left( \frac{2m}{m-r} \right) \cos 2rA \]  

we can write

\[ L_{(p,m)}(\lambda) = \int_{0}^{\lambda} t^{2p} e^{-t^2} \cos^{2m} \left( \frac{\pi t}{\lambda} \right) dt \]

\[ = 2^{-2m} \left( \frac{2m}{m} \right) \int_{0}^{\lambda} t^{2p} e^{-t^2} dt + 2^{1-2m} \sum_{r=1}^{m} \frac{2m}{m-r} \sum_{j=0}^{\infty} \left( \frac{2m}{m-r} \right) \frac{(2j)!}{(2j)!} \]  

\[ = 2^{-2m-1} \left( \frac{2m}{m} \right) \Gamma(p+\frac{1}{2}, \lambda^2) + 2^{-2m} \sum_{r=1}^{m} \frac{2m}{m-r} \sum_{j=0}^{\infty} \left( \frac{2m}{m-r} \right) \frac{(2j)!}{(2j)!} \]  

\[ \gamma(p+j+\frac{1}{2}, \lambda^2) \]

where \( \gamma(a, \lambda) \) are the incomplete gamma functions, defined as

\[ \gamma(a, \lambda) = \int_{0}^{\lambda} e^{-t^2} t^{a-1} dt, \quad \Re(a) > 0. \]  

This process is also known as the method of steepest descent. Another alternative form to deal with \( L_{(p,m)}(\lambda) \) is to obtain a suitable recurrence formula. For example, integrating eq. (33) by parts we obtain

\[ (2p+1) L_{(p,m)}(\lambda) = e^{-\lambda^2} \lambda^{2p+1} + 2L_{(p+1,m)}(\lambda) + \frac{2m}{\lambda} \int_{0}^{\lambda} e^{-t^2} t^{2p+1} \]

\[ \cos^{2m-1} \left( \frac{\pi t}{\lambda} \right) \sin \left( \frac{\pi t}{\lambda} \right) dt. \]  

For \( m = 0 \), eq. (38) reduces to a known result:

\[ L_{(p+1,0)}(\lambda) = \frac{1}{2} \left[ (2p+1) L_{(p,0)}(\lambda) - \lambda^{2p+1} e^{-\lambda^2} \right]. \]  

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Further, we note that

\[ L_{(0,0)}(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^\lambda e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} \text{Erf}\cdot\lambda \]  

(40)

where \text{Erf}\cdot\lambda is the error function, defined as \[\text{Erf}\cdot\lambda = \frac{2}{\sqrt{\pi}} \int_0^\lambda e^{-t^2} \, dt\]  

(41)

An alternative recurrence relation for \( L_{(p,m)}(\lambda) \) can be derived in the following form:

\[ L_{(p,m)}(\lambda) = \left[ 2m^2 + \left( \frac{\lambda}{\pi} \right)^2 (4p+1) \right]^{-1} \left\{ \left( \frac{\lambda}{\pi} \right)^2 e^{-\lambda^2} \left( \lambda^2 p+1 - p\lambda^2 p-1 \right) + \left( \frac{\lambda}{\pi} \right)^2 \left[ p(2p-1)L_{(p-1,m)}(\lambda) + 2L_{(p+1,m)}(\lambda) \right] + m(2m-1)L_{(p,m-1)}(\lambda) \right\} \]  

(42)

For \( m = 0 \), eq. (42) reduces to

\[ (4p+1)L_{(p,0)}(\lambda) = \left( \lambda^2 p+1 - p\lambda^2 p-1 \right) e^{-\lambda^2} + p(2p-1)L_{(p-1,0)}(\lambda) + 2L_{(p+1,0)}(\lambda) \]  

(43)

4. SOME RESULTS FOR \( S_{\mu}(k,\nu) \)

1. The differencing technique for \( S_{\mu}(k,\nu) \)

We shall use the differencing technique mentioned in section 2 for the function \( f(\theta) \) given, for this case, by

\[ f(\theta) = \frac{\sin 2\nu \theta}{(1-\kappa^2 \cos \theta)^{\mu + \frac{1}{2}}} \]  

(44)

where \( f^*(\theta) \) is given by
Using the transformation \( \cos \theta = 1 - 2w \) and binomial expansion, we obtain after simplification

\[
\int_{\theta=0}^{\pi} f^*(\theta) d\theta = \int_{0}^{\pi} \sin^{2\nu} \theta \left( \frac{1}{2} (\mu + \frac{1}{2}) (k^2 \cos \theta) \right) + \frac{1}{2} (\mu + \frac{1}{2}) k^2 \cos^2 \theta \left( \mu + \frac{Z}{6} \right) (k^2 \cos \theta) \] \]

Using eq. (16) we have

\[
\int_{\theta=0}^{\pi} f^*(\theta) d\theta = 2^{\nu} e^{\frac{1}{2} \nu} (\mu + \frac{1}{2}) k^2 \Gamma(\nu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2}) \frac{\Gamma(\nu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu+1)} \phi(\nu + \frac{1}{2}, 2\nu+1, -2k^2 (\mu + \frac{1}{2}))
\]

\[
+ (\mu + \frac{1}{2}) e^{k^2 (\mu + \frac{Z}{6})} 2^{2\nu-1} k^4 \left\{ \frac{\Gamma(\nu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu+1)} \phi(\nu + \frac{1}{2}, 2\nu+1, -2k^2 (\mu + \frac{Z}{6}))
\]

\[
- 4 \frac{\Gamma(\nu + \frac{3}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu+2)} \phi(\nu + \frac{3}{2}, 2\nu+2, -2k^2 (\mu + \frac{Z}{6}))
\]

\[
+ 4 \frac{\Gamma(\nu + \frac{5}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu+3)} \phi(\nu + \frac{5}{2}, 2\nu+3, -2k^2 (\mu + \frac{Z}{6})) \right\}
\]

\[\equiv B(\nu, k, \mu) \]
\[
\int_{0}^{\pi} (f(\theta) - f^*(\theta)) \, d\theta = \int_{0}^{\pi} \frac{\sin 2\nu \theta}{(1 - k^2 \cos \theta)^{\mu + \frac{1}{2}}} \, d\theta - \int_{0}^{\pi} \sin 2\nu \theta \cos^2 \theta \, e^{(\mu + \frac{1}{2})(k^2 \cos \theta)} \, d\theta - \frac{1}{2} (\mu + \frac{1}{2}) k^2 \int_{0}^{\pi} \sin 2\nu \theta \cos^2 \theta \, e^{(\mu + \frac{1}{2})(k^2 \cos \theta)} \, d\theta.
\]

Let

\[
\int_{0}^{\pi} (f(\theta) - f^*(\theta)) \, d\theta = M_1 - M_2 - M_3
\]

where

\[
M_1 = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^{2n} \Gamma(n + \frac{1}{2})}{(2n)! \Gamma(\nu + n + 1)} k^{4n},
\]

A result given by Kalla\(^4\).

\[
M_2 = \int_{0}^{\pi} \frac{\sin 2\nu \theta}{(1 - k^2 \cos \theta)^{\mu + \frac{1}{2}}} \, d\theta = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^{2n}}{n!} k^{2n} \int_{0}^{\pi} \sin 2\nu \theta \cos^n \theta \, d\theta
\]

since

\[
\int_{0}^{\pi} \sin 2\nu \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{2\nu + 1}{2}\right) \Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{2\nu + n}{2} + 1\right)} \quad \text{for } r \text{ even}
\]

\[
= 0 \quad \text{for } r \text{ odd}
\]

Then

\[
M_2 = \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2})^{2n}}{(2n)!} k^{2n} \frac{\Gamma\left(\frac{2\nu + 1}{2}\right) \Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{2\nu + n}{2} + 1\right)} \Gamma\left(\frac{2\nu + 2n}{2} + 1\right)
\]

Similarly we have .
Then, we have

\[
\int_0^\pi (f_1(\theta) - f_2(\theta)) d\theta = \sum_{n=0}^\infty \frac{(\mu + \frac{1}{2}) \cdot 2^n \cdot \Gamma(n + \frac{1}{2}) \cdot \Gamma(n + \frac{3}{2})}{(2n)! \cdot \Gamma(n + \frac{3}{2})} \cdot k^{2n} \cdot \frac{\Gamma\left(\frac{2n+1}{2}\right) \cdot \Gamma\left(\frac{2n+2+1}{2}\right)}{\Gamma\left(\frac{2n+2+1}{2}\right) + 1} \tag{51}
\]

Therefore

\[
\int_0^\pi \frac{\sin 2\nu \theta}{(1 - k^2 \cos^2 \theta)^{1 + \frac{1}{2}}} = B(\nu, k, \mu) + N(\nu, k, \mu) \tag{53}
\]

Particular cases:

For \( \nu = 0 \) eq. (48) reduces to

\[
\int_0^\pi f_2(\theta) d\theta = \int_0^\pi \left[ (\mu + \frac{1}{2}) (k^2 \cos \theta) + \frac{1}{2} (\mu + \frac{1}{2}) (k^2 \cos \theta) \right] d\theta \]

\[
= \pi \left[ I_0 (\mu + \frac{1}{2}) k^2) + \frac{k^4}{4} (2\mu + 1) I_0 ((\mu + \frac{7}{6}) k^2) \right] \quad - \frac{k^2}{4} \frac{(2\mu+1)}{(\mu + \frac{7}{6})} I_1 ((\mu + \frac{7}{6}) k^2) \tag{54}
\]
We notice that eq. (54) is equal to eq. (24) and obtained in a similar way. For \( V = 0 \) and \( \mu = j \), an integer, eq. (52) reduces to

\[
\int_0^\pi (f(\theta) - f^*(\theta)) \, d\theta = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)}{(2n)!\Gamma(n+1)} \frac{k^{4n}}{(j + \frac{1}{2})^{2n}} - \left( j + \frac{1}{2} \right)^{2n} \frac{2^n}{(n+1)}
\]

\[
\frac{k^n}{2} \left( j + \frac{1}{2} \right)^{2n} \frac{2^n}{(n+1)}
\]

As

\[
\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\Gamma\left(\frac{1}{2}\right)}{n!2^{2n}},
\]

therefore eq. (55) becomes

\[
\frac{1}{\pi} \int_0^\pi \left[ f(\theta) - f^*(\theta) \right] \, d\theta = \sum_{n=0}^{\infty} \frac{\frac{k^{4n}}{(n!)^2}}{2^{2n}} \frac{(2j+4n)!}{(j+2n)!2^{4n}(2j)!} - \frac{(2j+1)^{2n}}{2n+1}
\]

\[
\frac{k^n}{2} \left( 2j+1 \right)^{2n} \frac{2^n}{6^{2n}} \frac{(2n+1)}{2(n+1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{\frac{k^{4n}}{(n!)^2}}{2^{6n}} \left\{ \frac{(2j+4n)!}{(j+2n)!2^{4n}(2j)!} - \frac{2^n}{(2j+1)^{2n}} \right\}
\]

\[
- \sum_{n=1}^{\infty} \frac{(2j+1)18}{3} \frac{2^{2n}}{6^{2n}} \frac{(6j+7)^2}{2^{n+1}2^{n}} \frac{n(2n-1)}{6^{2n}(n!)^2}
\]

\[
= \sum_{n=2}^{\infty} \frac{\frac{k^{4n}}{(n!)^2}}{2^{6n}} \left\{ \frac{(2j+4n)!}{(j+2n)!2^{4n}(2j)!} - \frac{2^n}{(2j+1)^{2n}} \right\}
\]

\[-18 \left( \frac{2}{3} \right)^{2n} \frac{(2j+1)(6j+7)^{2n-2}}{n(2n-1)}
\]

We notice that eq. (56) is equal to eq. (27), which is given by Epstein and Hubbell².
II. $S_{\mu}(k,\nu)$ in terms of the Incomplete Gamma Function

$$S_{\mu}(k,\nu) = \int_0^{\pi} \sin^{2\nu} \theta \ \frac{\sin^{2\nu} \theta}{(1-k^2\cos \theta)^{\mu+\frac{1}{2}}}$$

$0 \leq k < 1, \ \text{Re}(\mu) > -\frac{1}{2}, \ \text{Re}(\nu) > -\frac{1}{2}$.

Let

$$S_{\mu}(k,\nu) = \int_0^{\pi} \sin^{2\nu} \theta \ e^{-F(\theta)} \ d\theta \quad (57)$$

where

$$F(\theta) = (\mu + \frac{1}{2}) \log(1-k^2\cos \theta) \quad (58)$$

Expanding $F(0)$ in Maclaurin series, we observe that the derivatives of odd order vanish and hence

$$S_{\mu}(k,\nu) = \int_0^{\pi} \sin^{2\nu} \theta \ e^{-F(\theta)} \ d\theta \quad (59)$$

Let

$$\left(\frac{\lambda}{\pi}\right)^2 = \frac{1}{2!} F^{(2)}(0)$$

$$S_{\mu}(k,\nu) = e^{-F(0)} \int_0^{\pi} \sin^{2\nu} \theta \ e^{-\left(\frac{\lambda}{\pi}\right)^2 \theta^2 \left[1 - \frac{F^{(4)}(0)}{4!} \theta^4 - \frac{F^{(6)}(0)}{6!} \theta^6 - \ldots\right]} \ d\theta \quad (60)$$

Now

$$F(0) = (\mu + \frac{1}{2}) \log(1-k^2)$$

$$t = \frac{\lambda}{\pi} \theta; \ dt = \frac{\lambda}{\pi} \ d\theta$$

$$S_{\mu}(k,\nu) = (1-k^2)^{-\frac{\mu}{2}} \left(\frac{\lambda}{\pi}\right)^{\frac{\mu}{2}} \int_0^{\frac{\lambda}{\pi} \theta} \sin^{2\nu} \left(\frac{\pi t}{\lambda}\right) e^{-t^2} \left[1 - \frac{F^{(4)}(0)}{4!} \left(\frac{\pi t}{\lambda}\right)^4 \theta^4 - \ldots\right] \ dt \quad (61)$$
Let

\[ L_{(p,v)}(\lambda) = \int_0^\lambda \frac{\sin^{2v} \left( \frac{\pi t}{\lambda} \right)}{t^{2p+1}} e^{-t^2} dt \]  

then

\[ S_\mu(k,v) = (1-k^2)^{-\mu-\frac{1}{2}} \frac{\pi}{\lambda} \left[ L_{(0,v)}(\lambda) - \frac{F(4)(0)}{4!} \left( \frac{\pi}{\lambda} \right)^4 L_{(2,v)}(\lambda) \right. \]

\[ \left. - \frac{F(6)(0)}{6!} \left( \frac{\pi}{\lambda} \right)^6 L_{(3,v)}(\lambda) \right] \]  

for \( v \) is an integer, and using

\[ \sin^{2v} = \frac{1}{2} \left( \binom{2v}{v} \right) + \frac{1}{2} \left( \binom{2v}{v-1} \right) \sum_{r=1}^{v} (-1)^r \left( \frac{2v}{v-r} \right) \cos 2\pi r \]  

we have

\[ L_{(p,v)}(\lambda) = \int_0^\lambda t^{2p} e^{-t^2} \left[ \frac{1}{2} \left( \binom{2v}{v} \right) + \frac{1}{2} \left( \binom{2v}{v-1} \right) \sum_{r=1}^{v} (-1)^r \left( \frac{2v}{v-r} \right) \cos \left( \frac{2\pi r t}{\lambda} \right) \right] dt \]

\[ = \frac{1}{2^{2v}} \left( \binom{2v}{v} \right) \gamma \left( p + \frac{1}{2} ; \lambda^2 \right) + \frac{1}{2^{2v}} \sum_{r=1}^{v} (-1)^r \left( \frac{2v}{v-r} \right) \sum_{j=0}^{\infty} \left( \frac{2\pi r}{\lambda} \right)^{2j} \frac{1}{(2j)!} \gamma \left( p + j + \frac{1}{2} ; \lambda^2 \right) \]

Particular cases:

\[ L_{(p,0)}(\lambda) = \frac{1}{2} \gamma \left( p + \frac{1}{2} ; \lambda^2 \right) \]

\[ L_{(p,1)}(\lambda) = \frac{1}{2} \gamma \left( p + \frac{1}{2} ; \lambda^2 \right) - \frac{1}{2} \sum_{j=0}^{\infty} \left( \frac{2\pi j}{\lambda} \right)^{2j} \frac{1}{(2j)!} \gamma \left( p + j + \frac{1}{2} ; \lambda^2 \right) \]
\( F^{(2)}(0), \ F^{(4)}(0) \) and \( F^{(6)}(0) \) given above are

\[
F^{(2)}(0) = (\mu + \frac{1}{2})k^2(1-k^2)^{-1}
\]
\[
F^{(4)}(0) = -(\mu + \frac{1}{2})k^2(1-k^2)^{-2}(2k^2+1)
\]
\[
F^{(6)}(0) = (\mu + \frac{1}{2})k^2(1-k^2)^{-3}(16k^4+13k^2+1)
\]

and

\[
\lambda = \frac{\pi k}{2} \sqrt{\frac{2\mu+1}{1-k^2}}.
\]

5. COMPUTATIONS

In table 1, we have computed the series expansions of \( K_\mu(k,m) \) which are given by eq.(11) and eq.(12) of ref.1, these equations are

\[
K_\mu(k,m) = \sum_{r=0}^{\infty} \tilde{H}_r(\mu,m) k^{4r}
\]

and

\[
\tilde{H}_r(\mu,m) = \frac{(\mu + \frac{1}{2})2r\pi \Gamma(m+r+\frac{1}{2})}{(2r)! \Gamma(m+r+1)}
\]

In fact, eq.(68) and eq.(69) were tabulated in table 1 of ref.1, but we have computed it again due to a small error in the computations of some values of \( K_\nu(k,m) \). The numerical integration of eq.(1), by using the trapezoidal rule, is represented by \( KT \).

In table 2, we have computed the series expansion of \( S_\mu(k,\nu) \) which is given by eq.(4) of ref.4:

\[
S_\mu(k,\nu) = \sum_{r=0}^{\infty} \frac{(\mu + \frac{1}{2})2r\Gamma(\nu + \frac{1}{2})\Gamma(r + \frac{1}{2})}{(2r)! \Gamma(\nu+r+1)} k^{4r}
\]

Equation (4) of ref.4 should be corrected to eq.(70), because the factor \( \Gamma(\nu + \frac{1}{2})^2 \) was typed instead of \( \Gamma(\nu + \frac{1}{2}) \).

The approximate asymptotic formula of \( K_\mu(k,m) \) in the neighbourhood of \( k=1 \) is obtained in simpler form and is given by
Table 1 - Values of $K_\mu(k,m)$ be series expansion eqr. (68), (69) and numerical integration of eq. (1) denoted by (K1).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>$\mu$</th>
<th>$K_\mu(k,m)$</th>
<th>K1</th>
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<td>3.1416515</td>
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<td>3.2191075</td>
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<td>1.5719152</td>
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<td>1.0</td>
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<td>7.5</td>
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</tr>
</tbody>
</table>

\[
K_\mu(k,m) = \sum_{r=0}^{2m} \frac{(-1)^r 2^r (2m)^r}{(1-k^2)^{\mu+1/2}} \frac{(\mu-r-1/2)!}{\lambda' r^r \Gamma(\mu+1/2-1/2\lambda')}
\]  

(71)

where $\lambda' = 2k^2/1 - k^2$.

This formula is much better than that one given by eq. (24) of ref.1, see table 3 of ref.1.

In table 3, eq. (71) is computed and is represented by $K3$.

We have also computed the asymptotic formula for $K_\mu(k,m)$ in the neighbourhood of $k = 1$ which is given by eq. (31) of ref.1.
Table 2 - Values of $S_{\nu}(k,v)$ for $v = 0,1$ and some selected values of $\mu$ and $k^2$ by using series expansion eq. (70).

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0.01</td>
<td>3.141897</td>
</tr>
<tr>
<td>0.02</td>
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<td>0.03</td>
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<td>3.701079</td>
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</tr>
<tr>
<td>0.50</td>
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</tbody>
</table>

Table 3 - Tabulation of $K_{\mu}(k,m)$ by using eq. (71) denoted by K3 and numerical integration (KI).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>$\mu$</th>
<th>K3</th>
<th>KI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
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<td>0.1270754</td>
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<tr>
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<tr>
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<td>8.6</td>
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</tbody>
</table>

153
This formula is valid for $\nu_r$ (positive integer) = $m - r$.

In Table 4 have computed eq. (72) which is represented by $K_4$. It is compared with $K_3$ and $K_1$.

Table 4 - Tabulation of $K_{\nu}(k,m)$ by using eq. (72) denoted by $K_4$, compared with $K_3$ and $(K_1)$.

<table>
<thead>
<tr>
<th>$k$</th>
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<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_1$</th>
</tr>
</thead>
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<td>0.18013358D 14</td>
<td>0.18013333D 14</td>
</tr>
<tr>
<td>0.999</td>
<td>0.2</td>
<td>0.236600D 06</td>
<td>0.236600D 06</td>
<td>0.236600D 06</td>
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<td>0.176551136D 22</td>
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Results were computed on a VAX/VMS electronic computing machine.

This work is supported by Kuwait University through a research grant No. SDM 119.

REFERENCES


Resumo