Equations of Motion of a Spinning Nucleon in a Scalar Field

N.K. SATO and S. RAGUSA
Departamento de Física e Ciências dos Materiais, Instituto de Física e Química de São Carlos, USP, Caixa Postal 369, São Carlos, 13560, SP, Brasil

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Abstract The equations of motion of a spinning nucleon interacting with a scalar massless field are derived from energy-momentum conservation and charge conservation following the moment method of Papapetrou.

1. INTRODUCTION

The equation of motion of a particle, which will be called nucleon, interacting with a scalar field has been derived by Harish-Chandra using the method that Dirac developed for the electron, by calculating the flux of the energy-momentum tensor and of the angular momentum tensor across a tube containing the world line of the particle. As in the case of Dirac, simplicity arguments were used to justify the contribution of the basis elements of the tube. In this paper we will derive the equation of motion by the method of moments of the energy-momentum tensor. The method of moments was introduced by Fock and elaborated by Papapetrou to describe the motion of a particle with spin in a gravitational field. This method was recently used to obtain the equations of motion of a non-abelian charged spin particle in a Yang-Mills field.

Besides the fact that the equations of motion can be obtained in a straightforward and simple way, the method of moments allows one to obtain the expression of the energy-momentum tensor and of the nucleonic current of the particle and of their moments.

2. THE EQUATION OF MOTION

The energy-momentum tensor of the scalar massless field $\phi$ is well-known.

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where we use the notation $\Phi^\alpha = \partial \phi / \partial x^\alpha = \partial^\alpha \phi$.

Calling $\tilde{\rho} = dg/d\tilde{V}$ the rest density of nucleonic charge, \( \frac{dg}{dV} \) being the quantity of nucleonic charge in the element of volume \( d\tilde{V} \) in the charge rest frame, the equation obeyed by the $\Phi$ field is

$$\partial^\alpha \partial_\alpha \Phi = -4\pi \tilde{\rho} \cdot$$

Note that as $\Phi$ is a Lorentz scalar the appropriate density is the scalar, actually invariant scalar, $\tilde{\rho}$. The density $\rho$ of the moving element with velocity $\tilde{v}$ is, in the system of units $c=1$

$$\rho = \frac{dg}{dV} = \tilde{\rho}(1-v^2)^{-1/2} = \rho u^0 = \tilde{\rho} \frac{dt}{d\sigma},$$

where $u^0$ is the time component of the four-velocity.

Using eq. (2) we can calculate the divergence of eq. (1) with the result

$$\partial_\beta T^{\alpha \beta}_{\tilde{F}} = -\tilde{\rho} \Phi^\alpha.$$

Now we introduce the nucleonic energy-momentum tensor $T^{\alpha \beta}$ by requiring energy-momentum conservation of the particle plus field system, which has the covariant expression

$$\partial_\beta (T^{\alpha \beta}_{\tilde{F}} + T^{\alpha \beta}) = 0 \cdot$$

From eq. (4) we see that $T^{\alpha \beta}$ satisfies the following equation:

$$\partial_\beta T^{\alpha \beta} = \tilde{\rho} \Phi^\alpha.$$

This equation is our starting point to obtain the equations of motion of the particle and of $T^{\alpha \beta}$ itself. The only thing that we demand is that $T^{\alpha \beta}$ be symmetric as $T^{\alpha \beta}_{\tilde{F}}$ is.

We shall consider our system as an extended body occupying a volume $V$, which will tend to zero at the end of the calculations to de-
scribe our point particle. Following Papapetrou we start by integration eq. (6) over $V$ to obtain

$$
\int (\partial_t T^{\alpha\sigma} + \partial_{\sigma} T^{\alpha\sigma}) \, dV = \int \bar{\rho} \phi^\alpha \, dV.
$$

(7)

By Gauss’s theorem the three-divergence term can be converted into a surface term, which is zero just outside the system, and we obtain the result

$$
\frac{d}{dt} \int T^{\alpha\sigma} \, dV = \int \bar{\rho} \phi^\alpha \, dV.
$$

(8)

Now, we choose a reference point $x^\alpha$ inside the system, which can be its center of mass as will be discussed later. Calling $\delta x^\alpha$ the distance of $x^\alpha$ to $x^\alpha$ we have

$$
x^\alpha = x^\alpha + \delta x^\alpha.
$$

(9)

This choice is being made at equal times, that is, $t = x^0$ or $\delta x^0 = 0$.

Next, we expand $\phi^\alpha(x)$ in eq. (8) around $x$ to obtain

$$
\phi^\alpha(x) = \phi^\alpha(x) + \delta x^\beta \partial_\beta \phi^\alpha(x) + \ldots
$$

(10)

We shall work first to zeroth order in $\delta x^\alpha$. To that order eq. (8) becomes, after using eq. (3),

$$
\frac{dP^\alpha}{ds} = g \phi^\alpha
$$

(11)

where $\phi^\alpha$ is now calculated at point $x^\alpha$,

$$
P^\alpha = \int T^{\alpha\sigma} \, dV
$$

(12)

is the momentum of our system and

$$
g = \int \rho \, dV
$$

(13)
its nucleonic charge. Of course this quantity is a constant as follows by integrating the equation of continuity for the nucleonic charge over the volume \( V \). We have

\[
\frac{\partial}{\partial t} \int_{V} J^{\alpha} dV = 0. \tag{14}
\]

By Gauss's theorem

\[
\frac{d}{dt} \int_{V} J^{0} dV = 0 = \frac{\partial}{\partial t} \int_{V} \rho dV. \tag{15}
\]

To obtain more information on \( \rho^{a} \) we consider the divergence of \( x^{\beta} T^{\alpha \lambda} \). With eq. (6) we obtain

\[
C_{\lambda} (x^{\beta} T^{\alpha \lambda}) = T^{\alpha \beta} + x^{\beta} \phi^{\alpha}. \tag{16}
\]

Integration over the volume of our system gives

\[
\frac{d}{dt} \int_{V} x^{\beta} T^{\alpha \lambda} dV = \int \frac{\partial}{\partial t} T^{\alpha \beta} dV + \int \rho \frac{\partial}{\partial t} x^{\beta} \phi^{\alpha} dV. \tag{17}
\]

Using eqs. (8) and (9) we obtain

\[
\frac{dx^{\beta}}{dt} \int T^{\alpha \lambda} dV + \frac{d}{dt} \int \hat{\phi}^{\alpha} x^{\beta} T^{\alpha \lambda} dV = \int T^{\alpha \beta} dV + \int \rho \hat{\phi}^{\alpha} x^{\beta} dV. \tag{18}
\]

To zeroth order in \( \hat{\phi}^{\beta} \) this equation becomes

\[
\frac{u^{\beta}}{u^{0}} \int T^{\alpha \lambda} dV = \int T^{\alpha \beta} dV \tag{19}
\]

where

\[
u^{\beta} \frac{dx^{\beta}}{ds} =. \tag{20}
\]

Putting \( a = 0 \) in eq. (19) we obtain with eq. (12)

\[
\rho^{\beta} = \frac{\rho^{0}}{u^{0}} u^{\beta}. \tag{21}
\]

Substituting this equation into equation (11) we have

\[
\frac{d}{ds} \left( \frac{\rho^{0}}{u^{0}} \right) u^{\alpha} + \frac{\rho^{0}}{u^{0}} \frac{du^{\alpha}}{ds} = g \phi^{\alpha}. \tag{22}
\]
Now we contract this equation with $u^\alpha$. Noting that $u_\alpha u^\alpha = 1$, $u_\alpha d u^\alpha / d \sigma = 0$ and $u_\alpha \partial^\alpha \phi = \partial \phi / \partial \sigma$, we obtain

$$
\frac{d}{d \sigma} \left( \frac{\nu^0}{u^0} \right) = g \frac{\partial \phi}{\partial \sigma}.
$$

From this we obtain a constant of motion

$$
\frac{\nu^0}{u^0} - g \phi = \text{constant} = m
$$

which we identify with the mass of our particle. Returning to eq. (22) we finally obtain the explicit form of the particle equation of motion

$$
m \frac{d u^\alpha}{d \sigma} = g \partial^\alpha \phi - g \frac{d (\phi u^\alpha)}{d \sigma}.
$$

This is the known equation for a particle interacting with a scalar field, which can be derived from the lagrangian $L = -(m + g \phi) \times (u^0)^{-1}$. As it was mentioned before, the moment method also gives the form of the energy-momentum tensor of the particle. From eqs. (12), (19), (21) and (24) we have

$$
\int T^{\alpha \beta} dV = \frac{1}{u^0} \frac{1}{(m + g \phi) u^\alpha u^\beta}.
$$

This is the integrated form of the energy-momentum of our particle.

From this we obtain

$$
T^{\alpha \beta} = (\tilde{\sigma} + \tilde{\rho} \phi) u^\alpha u^\beta
$$

where $\tilde{\sigma}$ is the rest mass density of the particle and $\tilde{\rho}$ its charge density

$$
\sigma (\tilde{x}) = \frac{1}{u^0} m \delta (\tilde{x} - \tilde{\sigma}), \quad \rho (\tilde{x}) = \frac{1}{u^0} g \delta (\tilde{x} - \tilde{\rho}).
$$

3. THE EQUATIONS OF MOTION FOR A SPINNING PARTICLE

After illustrating the procedure for a spinless point particle we turn to the case of a spinning particle with a nucleonic dipole mo-
ment. For this we substitute eq. (10) into eqs. (8) and (18) and we keep terms to first order in $\delta x^\alpha$. From the first equation we obtain

$$\frac{d\phi^\alpha}{ds} = g\phi^\alpha + D^\beta \partial_\beta \phi^\alpha \quad (29)$$

where

$$D^\alpha = \int \rho \delta x^\alpha \, dV \quad (30)$$

is the nucleonic dipole moment of our system.

From eq. (18) we obtain, after multiplication by $u^0$,

$$u^0 p^\alpha + \frac{d}{ds} \int \delta x^\alpha \, T^\alpha_0 \, dV = u^0 \int \delta x^\alpha \, T^\alpha_0 \, dV + D^\alpha \phi^\alpha \quad (31)$$

Subtracting from this equation the one obtained by interchanging $\alpha$ and $\beta$ we obtain the equation

$$\frac{d\delta x^\alpha}{ds} = p^\alpha u^\beta - p^\beta u^\alpha + D^\alpha \phi^\beta - D^\beta \phi^\alpha \quad (32)$$

where

$$S^{\alpha\beta} = \int (\delta x^\alpha \, T^{\beta_0} - \delta x^\beta \, T^{\alpha_0}) \, dV \quad (33)$$

is the spin tensor of the particle.

Eqs. (29) and (32) are the equations of motion of a nucleon with spin and dipole moment. They coincide with those obtained by Harish-Chandra by the method of Dirac. These equations are approximate equations of motion for an extended system and are exact for a point particle with spin and dipole moment, defined as the limit of the extended system when the volume or $\delta x^\alpha$ tend to zero, with $p$ and $S^{\alpha\beta}$ going to infinity in such a way that the integrals (12) and (13) remain finite, just as one defines the usual electric charge multipoles.

Considering that $g$ and $D^\alpha$ are given we have fourteen quantities to be determined, that is $p^\alpha$, $u^\alpha$ and $S^{\alpha\beta}$. As eqs. (29) and (32) add to only ten equations we still need four more equations to solve for the unknown quantities. These extra subsidiary equations are related to the choice of the particular point $X$ for the extended system. One possi-
bility is to choose $S^{\alpha 0} = 0$ in the system's rest frame. From eq. (33) this choice gives the following expression for the reference point

$$X^{\alpha} = \frac{\int x^{\alpha} T^{00} dV}{\int T^{00} dV}. \quad (34)$$

As we see the choice $S^{\alpha 0} = 0$ is equivalent to the choice of $X^{\alpha}$ as the center of energy of the extended system. In an arbitrary system the choice $S^{\alpha 0} = 0$ in the rest system becomes

$$u_{\alpha} S^{\alpha 0} = 0 \quad (35)$$

This gives the four additional equations that we need to specify our problem.

As we shall see now, we can also determine the energy-momentum tensors and the nucleonic current and their moments. Using eq. (33) we see that eq. (31) can be written as

$$u^{\alpha} \int T^{\alpha \beta} dV = u^{\beta} p^{\alpha} - D^{\beta} \phi^{\alpha} - \frac{1}{2} \frac{d S^{\alpha \beta}}{d s} + \frac{1}{2} \frac{d}{d s} \int (\delta x^{\beta} T^{00} + \delta x^{\alpha} T^{\beta 0}) dV. \quad (36)$$

To determine the last term in terms of $u^{\alpha}$ and $S^{\alpha \beta}$ we consider the divergence of $x^{\alpha} x^{\beta} T^{\lambda \nu}$

$$\nabla_{\nu}(x^{\alpha} x^{\beta} T^{\lambda \nu}) = x^{\beta} T^{\lambda \alpha} + x^{\alpha} T^{\lambda \beta} + x^{\alpha} x^{\beta} \rho_{\lambda}^{\nu}. \quad (37)$$

Integration of this equation over the system gives

$$\frac{d}{d t} \int x^{\alpha} x^{\beta} T^{\lambda 0} dV = \int (x^{\beta} T^{\lambda \alpha} + x^{\alpha} T^{\lambda \beta}) dV + \int x^{\alpha} x^{\beta} \rho_{\lambda}^{\nu} dV. \quad (38)$$

Using eqs. (9), (17) and (18) we obtain to order $\delta x^{\alpha}$,

$$\frac{d x^{\alpha}}{d t} \int \delta x^{\beta} T^{\lambda 0} dV + \frac{d x^{\beta}}{d t} \int \delta x^{\alpha} T^{\lambda 0} dV = \int (\delta x^{\alpha} T^{\lambda \beta} + \delta x^{\beta} T^{\lambda \alpha}) dV. \quad (39)$$

Now we add to this equation the one resulting from the interchange of $a$ and $\Lambda$ and subtract the one coming from the interchange of $\beta$ and $\Lambda$. The result is
\[ 2 \int \delta x \beta^\alpha \lambda dV = \frac{dx}{dt} \beta^\alpha + \frac{d\lambda}{dt} \delta^\alpha + \frac{d\beta}{dt} \delta^\lambda + \int (\delta x \beta^\alpha \lambda^0 + \delta x \lambda^0 \beta^\alpha) dV. \quad (40) \]

Putting \( \beta = 0 \) in this equation and recalling that \( \delta x^0 = 0 \) we obtain

\[ 0 = \frac{dx}{dt} S^0 \alpha^\lambda + \frac{d\lambda}{dt} \delta^\alpha + \int (\delta x \alpha^\lambda \lambda^0 + \delta x \lambda^0 \alpha^\lambda) dV. \quad (41) \]

Using this equation in (36) we obtain

\[ u^0 \int T^\alpha \beta dV = u^\beta \alpha^\alpha - B^\beta \delta^\alpha - \frac{1}{2} \frac{dS^\alpha \beta}{ds} + \frac{1}{2} \frac{d}{ds} \left( \frac{u^\beta}{u^0} S^0 \alpha^\alpha + \frac{u^\alpha}{u^0} S^\beta \lambda^0 \right). \quad (42) \]

This is the integrated form of the energy-momentum tensor of the system.

Using eqs. (41) in (40) we also obtain the integrated first moment of the energy-momentum tensor

\[ \int \delta x \beta^\alpha \lambda^0 dV = - \frac{1}{2} \left( \frac{u^\lambda}{u^0} S^0 \alpha^\alpha + \frac{u^\alpha}{u^0} S^\lambda \lambda^0 \right) + \frac{1}{2} \frac{d}{ds} \left( \frac{u^\beta}{u^0} S^0 \alpha^\alpha + \frac{u^\alpha}{u^0} S^\beta \lambda^0 \right). \quad (43) \]

Now we turn to the nucleonic current. With eq. (14) we have the identity

\[ \partial_\lambda (x^0 \gamma^\lambda) = f^\alpha. \quad (44) \]

Integration of this equation over the volume of our system gives

\[ \frac{d}{dt} \int x^\alpha J^0 dV = \int f^\alpha dV. \quad (45) \]

Using eqs. (9) and (13) we obtain

\[ \frac{dx}{dt} \gamma + \frac{d}{dt} \int \delta x \alpha \gamma^0 dV = \int f^\alpha dV. \quad (46) \]

Therefore

\[ \int f^\alpha dV = g \frac{x^\alpha}{u^0} + \frac{1}{u^0} \frac{dx}{ds}. \quad (47) \]
which is the integrated form of the current. Besides the convection term we also have a contribution from the dipole moment.

4 EQUATIONS OF MOTION FOR CONSTANT SPIN AND DIPOLE MOMENT

Let us now see in what physical circumstances we can simplify the equations of motion.

First we note that as $D^0 = 0$ we have

$$\dot{u}_\alpha D^\alpha = 0 \quad (48)$$

since this is valid in the particle rest frame. Also, as $dD^0/ds = 0$, we also have

$$\dot{u}_\alpha b^\alpha = 0 \quad (49)$$

where the dot designates $d/ds$.

From eqs. (48) and (49) it follows that

$$\dot{u}_\alpha D^\alpha = 0 . \quad (50)$$

From eq. (35) we have

$$u_\alpha b^\alpha = - \dot{u}_\alpha S^\alpha . \quad (51)$$

Contracting eq. (32) with $u_\beta$ and using eqs. (48) and (51) we have the relation

$$p^\alpha = (u_\beta p^\beta) u^\alpha - D^\alpha + \dot{u}_\beta S^\beta . \quad (52)$$

We shall now try to obtain $(u_\alpha p^\alpha)$. Contracting eq. (52) with $\dot{u}_\alpha$ and using eq. (50) together with the antisymmetric character of $S^\alpha$, we obtain

$$\dot{u}_\alpha p^\alpha = 0 . \quad (53)$$

Contraction of eq. (29) with $u_\alpha$ gives
Adding these last two equations we obtain

\[ \frac{d}{ds} (u_\beta p^\beta - g \phi - D^\beta \phi_\beta) = - \dot{\phi}^\beta \phi_\beta . \]  

(55)

As we shall see now, the right-hand side of this equation is zero if we postulate that both the spin and the dipole moment of the particle have constant magnitudes, that is

\[ S^\alpha_\beta S_\alpha^\beta = \text{constant} \]  

(56)

\[ D_\alpha^{'} D_\alpha = \text{constant} \]  

(57)

Contracting eq. (32) with \( S_\alpha^\beta \) and using eqs. (35) and (56) we have \( S_\alpha^\beta D^\beta \phi_\alpha = 0 \) and, therefore, as \( \phi_\alpha \) is arbitrary

\[ S_\alpha^\beta D^\beta = 0 \]  

(58)

From this we have

\[ S_\alpha^\beta D^\beta = - S_\alpha^\beta \dot{\phi}^\beta \]  

(59)

Now contract eq. (32) with \( D_\alpha^\beta \phi_\beta \). Using eq. (59) and the antisymmetric nature of \( S^\alpha_\beta \) together with eqs. (48), (49) and (57) we obtain the relation

\[ 0 = D_\alpha^\beta \phi_\alpha D_\beta \phi_\beta \]  

(60)

From here it follows that

\[ \phi_\alpha D_\beta \phi_\beta = 0 \]  

(61)

In this way eq. (55) gives us the following constant of motion:

\[ u_\beta p^\beta - g \phi - D^\beta \phi_\beta = \text{constant} = m \]  

(62)

which we identify with the mass of particle.
From this we obtain
\[ u_\beta p^\beta = m + g \phi + D^\alpha \phi_\alpha . \]  
(63)

Substituting this result into eq. (52) we obtain
\[ p^\alpha = (m + g \phi + D^\beta \phi_\beta) u^\alpha - D^\alpha \phi + \dot{u}_\beta s^{\beta \alpha} . \]  
(64)

From eqs. (35) and (58) we see that spin tensor is perpendicular to both the four-velocity and the dipole moment, that is, we can write the relation
\[ S^{\alpha \beta} = I \ v^\alpha v^\beta u_\lambda \partial_\nu \]  
(65)

Contracting this equation with itself and using eq. (48) we obtain
\[ S^{\alpha \beta} S_{\alpha \beta} = -2 I^2 D^\lambda D_\lambda \]  
(66)

This shows that \( I \) is a constant and that the magnitude of the spin and the dipole moment are related to each other.

Eqs. (64) and (65) coincide with the expression given by Harish-Chandra who uses eq. (65) to prove eq. (64).

Substituting eq. (65) into (64) we obtain
\[ p^\alpha = (m + g \phi + D^\beta \phi_\beta) u^\alpha - D^\alpha \phi + I \ v^\alpha v^\beta u_\lambda \dot{u}_\beta \partial_\nu \]  
(67)

Using this result in eq. (29) we obtain
\[ \frac{\partial}{\partial \sigma} \left[ (m + g \phi + D^\beta \phi_\beta) u^\alpha - D^\alpha \phi + I \ v^\alpha v^\beta u_\lambda \dot{u}_\beta \partial_\nu \right] = g \phi^\alpha + p^\beta \partial_\beta \phi^\alpha . \]  
(68)

Substitute now eqs. (65) and (67) into eq. (32). After using the equality
\[ u_\lambda \ v^\nu u_\beta \ v^\delta = u_\beta \ v^\nu u_\lambda \ v^\delta \]  
which can be verified by contraction of the right-hand side with \( \epsilon^{\alpha \beta \gamma \delta} \), we obtain
Eqs. (68) and (69), which contain two arbitrary constants $I$ and $m$, are the equations of motion of the nucleon with spin and dipole moment constant in magnitude.

REFERENCES


Resumo

As equações de movimento de um nucleon com spin em interação com um campo escalar sem massa são obtidas a partir da conservação da energia-momentum e da carga, pelo método dos momentos de Papapetrou.