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A Comment on the Proof of Noether's Theorem is Smooth Manifolds

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Abstract We give a proof of Noether's theorem in a smooth manifold which does not use the hypothesis that the one parameter family of diffeomorphisms of M, $\{h_s, s \in R\} = C$, which leaves the Lagrangian invariant, is a group. We also discuss the physical meaning of the restriction that C is an one-paraneter group of diffeomorphisms.

1. INTRODUCTION

Let M be a smooth* manifold, TM the tangent bundle of M, $L:TM \to R$ a smooth function. Let $h : M \to M$ be a diffeomorphism. We say that the pair (M,L) is invariant under h if

$$L(h(x),\ h_*(v_x)) = L(x,v_x)\ \forall\ (x,v_x) \in TM \tag{1}$$

In eq. (1) $h_* : TM \to TM$ is the derivative mapping of the rnap h. Let $\mathrm{Inv}(L)$ be the set of all diffeomorphisms of M such that for each $h \in \mathrm{Inv}(L)$ eq.(1) holds true. Let $S \subset \mathrm{Inv}(L)$. We say that the pair (M,L) is invariant under the set of diffeomorphisms S if eq.(1) is valid for each element in S. Note that whereas $\mathrm{Inv}(L)$ is a group, the set S is not in general a group. In particular, let $C:R \to \mathrm{Inv}(L)$ be a curve in the space of diffeomorphisms, such that $h_{s_1} = C(s_1)$ and $h_{s_2} = C(s_2)$. This curve will generate a one-parameter group of diffeomorphisms under the operation of composition iff

* In this paper *smooth* means differentiable of class C^k, with k such that the statements made are valid.

(i) h_s is differentiable

(ii) $h_0(x) = x \ \forall x \in M$

(iii) $h_{s_1} \circ h_{s_2} = h_{s_1+s_2} \ \forall s_1, s_2 \in \mathbb{R}$

When (i) and (ii) are satisfied but (iii) *is not satisfied* we call the curve $C \equiv \{h_s, s \in R\}$ a one-parameter family of diffeomorphisms of M (OPFDM).

The usual presentation of Noether's theorem[1,2] admits that the curve $C \subset \mathrm{Inv}(L)$ generates a one-parameter group of diffeomorphisms. In this paper we give in section 2 a proof of Noether's theorem where only the hypothesis that C is an OPFDM is used. In section 3 we discuss the physical meaning of the group hypothesis, using an example. We arrive at the conclusion that the more general hypothesis that C is a one-parameter group of diffeomorphisms does not give genuinely new conservation laws.

2. NOETHER'S THEOREM

Theorem: If the system (M,L) is invariant under a OPFDM, $C = h_s, s \in R$ then the Lagrangian system of equations corresponding to L has a first integral.

Proof: (A) First assume $M = R^n$. Let $x^i : R^n \to R$ be the canonical coordinate functions, and let $\phi : R \to R^n$ by $t \mapsto \phi(t)$ be a curve. Define the mapping $\bar{q} = \{x^i \circ \phi \equiv q^i, \frac{d}{dt} q^i \equiv \dot{q}^i, i = 1,\ldots,n\} \equiv \{q,\dot{q}\}$. Write $\bar{L} = L \circ \bar{q}$. Suppose now that the curve ϕ is a solution of the Euler-Lagrange equations, which we write as

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial q}\right) - \frac{\partial \bar{L}}{\partial q} = 0 \tag{2}$$

where $\partial\bar{L}/\partial q^i$ means $(\partial\bar{L}/\partial x^i) \circ \phi$, etc.

As L is supposed to be invariant under $h_s \in C$ the curve $h_s \circ \phi : R \to R^n$ is also a solution of the Euler-Lagrange equations for all $s \in R$. Define

$$\Phi : (s,t) \mapsto \Phi(s,t) = h_s(\phi(t)); \ \bar{Q} = \{x^i \circ \Phi \equiv Q^i, \frac{\partial}{\partial t} Q^i \equiv \dot{Q}^i, i=1,\ldots,n\}$$

and $\tilde{L} = L \circ \tilde{Q}$. We have, for fixed s,

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}}\right) - \frac{\partial \tilde{L}}{\partial Q} = 0 \qquad (3)$$

Since $h_s \in C$, we can write

$$L(Q(s,t), \frac{\partial Q}{\partial t}(s,t)) = L(q(t), \frac{d}{dt} q(t)) \qquad (4)$$

Eq. (4) implies that

$$\frac{\partial \tilde{L}}{\partial s} = 0$$

i.e.,

$$\frac{\partial \tilde{L}}{\partial Q}\frac{\partial Q}{\partial s} + \frac{\partial \tilde{L}}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial s} = 0 \quad , \quad \frac{\partial Q}{\partial s}\frac{\partial}{\partial Q} \equiv \sum_{i=1}^{n} \frac{\partial Q^i}{\partial s}\frac{\partial}{\partial Q^i} \quad ; \quad \text{etc} \qquad (5)$$

Using now eq. (3) into eq. (5) we have

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}}\right)\frac{\partial Q}{\partial s} + \frac{\partial \tilde{L}}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial s} = 0$$

then

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}}\ \frac{\partial Q}{\partial s}\right) = 0 \qquad (6)$$

and we get

$$\frac{\partial}{\partial \dot{Q}} L(Q(s,t), \frac{\partial}{\partial t} Q(s,t)) \frac{\partial}{\partial s} Q(s,t) = \text{constant} \qquad (7)$$

Eq. (7) must be valid for **all** $s \in R$. **In** the particular **case when** $s = 0$, **i.e.,** when $h_s = Id$ we obtain

$$I(q,\dot{q}) = \frac{\partial \tilde{L}}{\partial \dot{q}}\frac{d}{ds}(h_s(q))\bigg|_{s=0} \equiv \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i}(q^i,\dot{q}^i)\ \frac{d}{ds}(h_s^i(q))\bigg|_{s=0} = \text{constant} \qquad (8)$$

where $h_s^i(q) = x^i \circ h_s \circ \phi$, $i = 1,\dots,n$.

This concludes the first part of the proof. We must now **show that** the **same** result is obtained when M is an arbitrary smooth manifold.

(B) Let then $A = \{(U_j,\psi_j)\}_{j\in J}$ $(J \subseteq N)$ a **maximal** atlas of M. **It** follows that the set TA = $\{(TU_j, T\psi_j)\}_{j\in J}$ is an atlas for TM, where $TU_j = \pi^{-1}(U_j)$ and $T\psi_j : TU_j \to U'_j \times R^n$ is the tangent map of the $\psi_j : U_j \to U'_j \subset R^n$, and $\pi : TM \to M$ is the canonical projestion.

We now define $\bar{L} : R^{2n} \to R$ as the representation of L in the atlas TA, i.e., $\bar{L}(T\psi_j(q,\dot{q})) = L(q,\dot{q})\,\forall j \in J$. Since L is invariant under the action of the OPFOM $\{\bar{h}_s,\ s \in R\}$, $\bar{L}$ will be invariant under the acticn of the OPFDH $\{\bar{h}_s,\ s \in R\}$ where $h_s : R^n \to R^n$ is the representation of h_s in the atlas A, i.e., $\bar{h}_s = \psi_k \circ h_s \circ \psi_j^{-1}$ (see fig.1). Indeed, we must have

$$L(q,\dot{q}) = \bar{L}(T\psi_j(q,\dot{q}))$$

$$L(Th_s(q,\dot{q})) = \bar{L}(T\psi_k(Th_s(q,\dot{q})) \qquad (9)$$

where $Th_s : TM \to TM$ is the tangent mapping of h_s, $Th_s(q,\dot{q}) = (h_s(q), h_{s*}(q))$.

Since by hypothesis $L(q,\dot{q}) = L(Th_s(q,\dot{q})$ it follows that

$$\bar{L}(T\psi_j(q,\dot{q})) = \bar{L}(T\psi_k(Th_s(q,\dot{q})))$$

$$= \bar{L}(T\psi_k \circ Th_s(q,\dot{q}))$$

$$= \bar{L}(T\psi_k \circ Th_s \circ (T\psi_j)^{-1}(T\psi_j(q,\dot{q})))$$

However,

$$T\psi_k \circ Th_s \circ (T\psi_j)^{-1} = T\psi_k \circ Th_s \circ T(\psi_j^{-1})$$

$$= T(\psi_k \circ h_s \circ \psi_j^{-1}) = T\bar{h}_s$$

and then

$$\bar{L}(T\psi_j(q,\dot{q})) = \bar{L}(T\bar{h}_s(T\psi_j(q,\dot{q}))) \qquad (10)$$

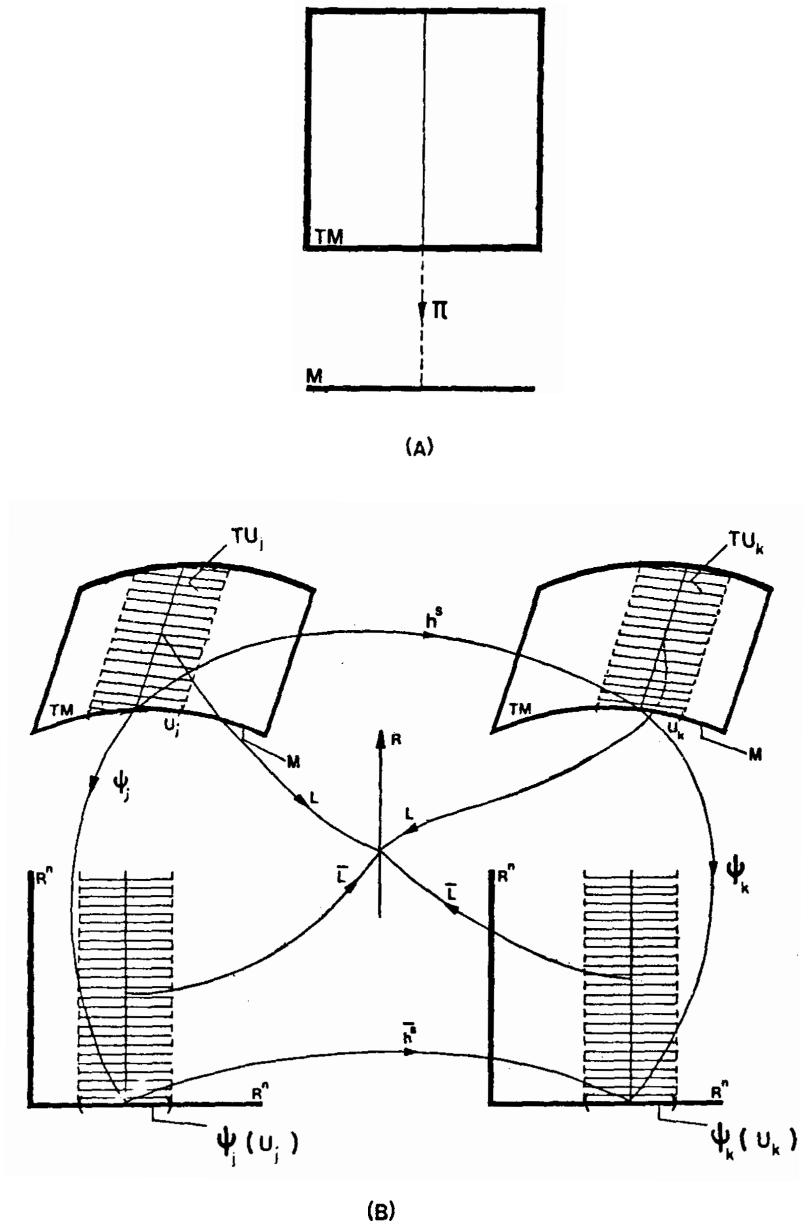

Fig.1 - Maps used in the proof of Noether's Theorem.

as required. Putting $T\psi_j(q,\dot{q}) = (\hat{q},\dot{\hat{q}})$, eq.(10) can be written as

$$\tilde{L}(\hat{q},\dot{\hat{q}}) = \bar{L}(T\bar{h}_s(\hat{q},\dot{\hat{q}})) \tag{10'}$$

Now the fact that the function $\bar{L} : R^{2n} \to R$ is invariant under the OPFDH, $\{\bar{h}_s, s \in R\}$ implies as already shown that the quantities

$$\bar{I}(\hat{q},\dot{\hat{q}}) = \frac{\partial\bar{L}}{\partial q}\frac{d}{ds}(\bar{h}_s(\hat{q}))\Big|_{s=0} = \text{constant} \tag{8'}$$

To complete the proof we must show that the result obtained does not depend on the atlas used to parametrize the manifold. But this point is indeed trivial, since we assumed A to be a maximal atlas.

3. THE PHVSICAL MEANING OF THE GROUP PROPERTY FOR THE OPFDM, $\{h_s, s \in R\}$

It may seem strange to the reader that we did not use at any point of the proof of Noether's theorem the fact that the OPFDM $\{h_s, s \in R\}$ must be a group. As is now clear all we need in the proof is that the family of diffeomorphisms can be parametrized by the set of real numbers. We then get the following question.

What motivates the usual assumption that $\{h_s, s \in R\}$ is an one-parameter group of diffeomorphisms? The answer, as will be made clear by an elementary example, is that we do not obtain results with more generality by imposing L to be invariant under a OPFOM. To convince ourselves that this is indeed the case consider the Lagrangian $L:TR^2 \to R$, by

$$TR^2 \ni (\vec{x},\dot{\vec{x}}) \to \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \in R$$

where m is a real (and positive) parameter and $\vec{x} = (x,y)$. As is well known, L represents the Lagrangian of a free particle in two dimensions. A set of natural conservation laws for this system is the one establishing the constancy of the linear momenta in the direction of the coordinate axes

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} = \text{constant}$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} = \text{constant} \tag{12}$$

Eqs. (12) are consequences of the invariance of the Lagrangian under translations along the x and y directions. More formally, thefacl: that L is invariant under the one-parameter group of diffeomorphisms

$$h^\alpha(x,y) = (x+\alpha,y) \quad ; \quad \frac{dh^\alpha}{d\alpha} = (1,0) \quad , \alpha \in R$$

i.e., $L(h^\alpha(x,y),\ h^\alpha_*(x,y)) = L(x,y),(\dot{x},\dot{y}))$ implies that the quantity $\frac{\partial L}{\partial \dot{\vec{x}}}\frac{dh^\alpha}{d\alpha} = m\dot{x}$ is a constant of motion. Analogously, since the Lagrangian is invariant under the one-parameter group of diffeomorphisms

$$h^\beta(x,y) = (x,\ y+\beta) \quad ; \quad \frac{dh^\beta}{d\beta} = (0,1) \quad , \beta \in R$$

the quantity $\frac{\partial L}{\partial \dot{\vec{x}}}\frac{dh^\beta}{d\beta} = m\dot{y}$ is conserved during the motion of the particle.

Now, from Noether's theorem we can get the result that the projection of the rnomentum vector is constant along an arbitrary direction in the plane. Indeed, it is enough to observe that the Lagrangian is invariant under the one-parameter group of diffeomorphisms

$$h^\alpha(x,y) = (x + a\alpha,\ y + b\alpha); \quad a,\ b \text{ are constants,} \quad \alpha \in R. \tag{13}$$

We then get that the quantity $m\dot{x}a$ + $m\dot{y}b$ is conserved. Now, let us consider, only an OPFDM instead of a group. Put

$$h^\alpha(x,y) = (x + f(\alpha),\ y + g(\alpha)), \quad \alpha \in R \tag{14}$$

where f and g are arbitrary C' functions. It is trivial to show that the Lagrangian, eq. (11) is invariant under the *translations* defined by eq. (14) and Noether's theorem gives as conserved quantity

$$m\dot{x}\frac{df}{d\alpha} + m\dot{y}\frac{dg}{d\alpha} \tag{15}$$

i.e., the projection of the linear momentum is conserved in the direction of the vector $(\frac{df}{d\alpha}, \frac{dg}{d\alpha})$.

Now, for each $a = \alpha_0$ (fixed) there exists a one-parameter group of diffeomorphisms that reproduce the above result. Indeed, it is enough in eq. (13) to put $a = \frac{df}{d\alpha}\Big|_{\alpha_0}$ and $b = \frac{dg}{d\alpha}\Big|_{\alpha_0}$. It follows that we do not obtain real novelties allowing for the set $\{h^a, a \in R\}$ a more general structure as an OPFDM.

We end this paper with two more comments:

(a) The admission of a more general structure than that of an OPFDM is not possible in more general physical applications. We must keep in mind that in the mathematical implementation of any theory, physical quantities are more conveniently described by geometrical objects defined on the manifold, which can be characterized by their transformation properties in relation to the group of diffeonorfisms of the manifold.

(b) We must also observe that in this paper we work only with one-parameter families and groups of diffeomorfisms. The treatment, can indeed, be generalized to the case where we have an r-parameter set of diffeomorfisms of the rnanifold. We should then substitute an *arbitrary* *r*-dimensional Lie group, with r greater than one, for the set **R** of real nurnbers which parameters the set of diffeormorfisms. In this way, using the same procedure as before, we shall arrive ar a set of conservation laws which we shall conclude to be the same as (or, at the worst, a linear combination of) the r conservation laws we should obtain by taking *r* one-dimensional Lie groups which *reproduce* (locally) the r-dimensional one.

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Resumo

Damos uma prova do teorema de Noether em uma variedade lisa M, a qual não usa a hipótese de que a família a um parâmetro de difeomorfismos de M, $\{h_s, s \in R\} = C$, que deixa a Lagrangeana invariante, seja um grupo. Também discutimos o significado físico da restrição de que C seja um grupo de difeomorfismos a um parâmetro.