On a Generalized Elliptic-Type Integral

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Abstract In this paper we study a family of integrals of the form

\[ K_{\mu}(k,m) = \int_0^{\pi} \frac{\cos^{2m} \theta}{(1 - k^2 \cos \theta)^{\mu + \frac{1}{2}}} d\theta \]

where \(0 < k < 1\), \(\text{Re}(\mu) > -\frac{1}{2}\) and \(m\) is a nonnegative integer. Such integrals occur in radiation field problems. We obtain a series expansion and establish its relationship with Gauss' hypergeometric function. Asymptotic expansions valid in the neighbourhood of \(k^2 = 1\) are given. One of these formulas has been obtained by the use of an Abelian theorem. Some recurrence relations are established. Results obtained earlier by Epstein and Hubbell, and Weiss follow as particular cases of our formulae given here. Some numerical values of \(K_{\mu}(k,m)\) for selected values of the parameter are tabulated, using different formulae.

1. INTRODUCTION

Epstein and Hubbell [1, p.1] have treated a family of integrals

\[ \Omega_j(k) = \int_0^{\pi} (1 - k^2 \cos \theta)^{-j - \frac{1}{2}} d\theta \]

where \(0 \leq k < 1\). Such integrals are found in the application of the Legendre polynomial expansion method [2, p.109] to certain problems involving computation of the radiation field off-axis from a uniform circular disc radiating according to an arbitrary distribution law [3, p.249].

In continuation of the Epstein-Hubbell work, Weiss [4, p.1] has obtained an expansion of (1) in the neighborhood of \(k^2 = 1\) and established its relationship with Legendre and hypergeometric functions.

Recently, Kalla and Kalla, Conde and Hubbell have defined and studied certain generalized elliptic-type integrals.

In the present work, we study the family of integrals

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where \( 0 \leq k < 1, \text{Re}(\mu) > -\frac{1}{2} \) and \( m \) is a non-negative integer. We observe that for \( m=0 \) and \( \mu=j \), a positive integer

\[
K_j(k,0) = \Omega_j(k)
\]

and further

\[
K_0(k,0) = \Omega_0(k) = (\sqrt{\lambda}/k)K(\lambda)
\]

\[
K_1(k,0) = \Omega_1(k) = (\sqrt{\lambda}/k(1-k^2))E(\lambda)
\]

where

\[
\lambda^2 = 2k^2/(1 + k^2)
\]

and

\[
K(\lambda) = \int_0^{(\pi/2)} (1 - \lambda^2 \sin^2 \theta)^{-\frac{1}{2}} d\theta
\]

\[
E(\lambda) = \int_0^{(\pi/2)} (1 - \lambda^2 \sin^2 \theta)^{\frac{1}{2}} d\theta
\]

are the complete elliptic integrals of the first and second kind respectively [7,p.295; 8,p.587].

First, we obtain a series expansion of \( K_{11}(k,m) \) for small values of \( k \). We establish a relation between our generalized elliptic-type integral and Gauss' hypergeometric function. Asymptotic expansions of \( K_{11}(k,m) \), valid in the neighborhood of \( k^2=1 \) are obtained. One of the results has been obtained by the use of the transformation formulae for the hypergeometric function, while the other, by an appeal to an Abel theorem. Some recurrence formulae are established.

2. SERIES EXPANSION

We have
where \( 0 \leq k < 1 \), \( \Re(\mu) > -\frac{1}{2} \) and \( m \) is a non-negative integer. We use the binomial expansion for small values of \( k \) to obtain

\[
K_\mu(k,m) = \sum_{r=0}^{\infty} \frac{k^{2r} (\mu + \frac{1}{2})_r}{r!} \int_0^{\pi} \cos^{2m+2r} \theta \, d\theta
\]

Evaluating the \( 8 \)-integral, by using

\[\int_0^{\pi} \cos^n \theta \, d\theta = \left[ 1 + (-1)^n \right] (\pi/2)^{n/2} \frac{\Gamma(n/2 + \frac{1}{2})}{\Gamma(n/2 + 2)}\]

we get the required expansion

\[
K_\mu(k,m) = \sum_{r=0}^{\infty} \mathcal{W}_r(\mu,m) k^{4r}
\]

where

\[
\mathcal{W}_r(\mu,m) = \frac{(\mu + \frac{1}{2})_2}{(2r)!} \frac{\pi^{r+1} \Gamma(m + r + \frac{1}{2})}{\Gamma(m + r + 1)}
\]

From (11), we can derive

\[
K_j(k, 0) = \sum_{r=0}^{\infty} \frac{k^{4r} \pi^{j+1} \Gamma(2j+4r+1)}{(64)^r \Gamma(2j+1) \Gamma(j+2r+1)}
\]

a result given earlier by Epstein and Hubbell \([1,p.3]\).

3. RELATION WITH GAUSS HYPERGEOMETRIC FUNCTION

We use the well known result

\[
\frac{1}{e^{u^{\frac{1}{2}}} \Gamma\left(\mu + \frac{1}{2}\right)} \int_0^{\infty} t^{\mu - \frac{1}{2}} e^{-st} \, dt, \Re(x) > 0, \Re(\mu) > -\frac{1}{2}
\]

to obtain
to obtain

\[ K_\mu(k, m) = \int_0^\infty \cos 2m_\theta \left[ -\frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\infty t^{\mu+\frac{1}{2}} e^{-(1-k^2\cos \theta)t} \, dt \right] \, d\theta \]

Interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved, we get

\[ K_\mu(k, m) = \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\infty e^{-t} t^{\mu+\frac{1}{2}} \left[ \int_0^\pi \cos 2m_\theta \, e^{k^2t} \, \cos \theta \, d\theta \right] \, dt \]  \[ (15) \]

Now, expressing \( \cos 2m_\theta \) as \( (1 - 2\sin^2 \frac{\theta}{2}) \, 2m_\theta \) and using the binomial expansion, the \( \theta \)-integral reduces to:

\[ \sum_{r=0}^{2m} (-1)^r 2^r \binom{2m}{r} \, e^{-k^2t} \int_0^1 u^{\frac{1}{2}} (1-u)^{\frac{1}{2}} \, e^{2k^2tu} \, du \]

This can be rewritten as,

\[ \sum_{r=0}^{2m} \frac{(-1)^r 2^r \binom{2m}{r} \, e^{-k^2t} \Gamma(r + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(r+1)} \Phi(\frac{1}{2}, r+1; 2k^2t) \]

by virtue of the integral representation for the confluent hypergeometric function \([9, p.255, 10, p.266]\)

\[ \Phi(\alpha, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma-\alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\gamma-\alpha-1} \, dt \] \[ (17) \]

Hence, we have

\[ K_\mu(k, m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} \frac{(-1)^r 2^r \binom{2m}{r} \Gamma(r+\frac{1}{2}) \Gamma(r+1)}{\Gamma(r+1)} \int_0^\infty e^{-(1+k^2)t} t^{\mu+\frac{1}{2}} \Phi(\frac{1}{2}, r+1; 2k^2t) \, dt \]
Evaluating the $x$-integral, we obtain

$$K_{\mu}(k,m) = \sum_{r=0}^{2m} (-1)^{2r} \binom{2m}{r} \frac{\Gamma\left(\frac{1}{2}ight) \Gamma\left(r+\frac{1}{2}\right)}{(2k^2)^{1+\frac{1}{2}} \Gamma(r+1)}$$

$$\int_{0}^{\infty} \frac{-(1+k^2)x}{2k^2} x^{1-\frac{1}{2} \phi\left(\frac{1}{2}, r+1; x\right) dx}$$

(18)

From (19), we deduce that

$$K_{\mu}(k,0) = \Omega_{\mu}(k) = \prod_{1}^{\pi} \frac{1}{(1+k^2)^{\frac{1}{2}}} 2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; 2k^2/(1+k^2)\right),$$

(20)

a result given by Weiss [4, p. 2 (9)].

4. ASYMPTOTIC EXPANSIONS

By recast (19) by using [9, p. 105 (1)]

$$K_{\mu}(k,m) = (1+k^2)^{-\mu-\frac{1}{2}} \sum_{r=0}^{2m} \frac{(-1)^{2r} \binom{2m}{r} \Gamma\left(\frac{1}{2}\right) \Gamma\left(r+\frac{1}{2}\right)}{\Gamma(r+1)}$$

$$\left[\frac{(1-k^2)^{2m}}{1+k^2}\right] 2F_1\left(r+\frac{1}{2}, r-\mu+\frac{1}{2}; r+1; 2k^2/(1+k^2)\right)$$

(21)

and then [9, p. 105 (3)] leads to

$$K_{\mu}(k,m) = (1-k^2)^{-\mu-\frac{1}{2}} \sum_{r=0}^{2m} \frac{(-1)^{2r} \binom{2m}{r} \Gamma\left(\frac{1}{2}\right) \Gamma\left(r+\frac{1}{2}\right)}{\Gamma(r+1)}$$

$$\left[\frac{(1+k^2)^{2m}}{1-k^2}\right] 2F_1\left(r+\frac{1}{2}, \mu+\frac{1}{2}; r+1; 2k^2/(k^2-1)\right)$$

(22)
Now the formula \([9, p. 107 (34)]\) can be used to obtain
\[
K_{\mu}(k, m) = \sum_{n=0}^{2m} \frac{(-1)^n 2^n \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + \frac{1}{2} \right)}{\Gamma(n + 1)}
\]
\[
\left( \frac{\Gamma(n - \mu) \Gamma(n + 1)}{\Gamma(n - \frac{1}{2}) \Gamma(n + \frac{1}{2})} \right) \frac{1}{(2k^2)^{n+\frac{1}{2}}} \left[ \frac{\Gamma(n - \mu) (1 - k^2)^{n-\mu}}{\Gamma(n - \frac{1}{2}) (2k^2)^{n+\frac{1}{2}}} + \frac{\Gamma(n - \mu) (2k^2)^{-\mu - \frac{1}{2}}}{\Gamma(n - \mu + \frac{1}{2}) \Gamma(n + \frac{1}{2})} \right]
\]
Hence an expansion formula valid in the neighborhood of \(k^2 = 1\), can be expresses as

\[
K_{\mu}(k, m) = \sum_{n=0}^{2m} \frac{(-1)^n 2^n \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + \frac{1}{2} \right)}{\Gamma(n + 1)}
\]
\[
\left( \frac{\Gamma(n - \mu) \Gamma(1 - k^2)^{n-\mu}}{\Gamma(n - \frac{1}{2}) (2k^2)^{n+\frac{1}{2}}} + \frac{\Gamma(n - \mu) (2k^2)^{-\mu - \frac{1}{2}}}{\Gamma(n - \mu + \frac{1}{2}) \Gamma(n + \frac{1}{2})} \right)
\]
A particular case of (24)

\[
K_{\mu}(k, 0) = \sqrt{\pi} \left[ \frac{\Gamma(n)(1 - k^2)^{-\mu}}{\Gamma(n + 1) (2k^2)^{\frac{1}{2}}} + \frac{\Gamma(n) (2k^2)^{-\mu - \frac{1}{2}}}{\Gamma(n + 1)} \right]
\]
(provided \(\mu\) is not an integer) is in agreement with a special case of a result given by Kalla, Conde and Hubbell\(^6\).

Now we shall obtain another asymptotic expansion valid in the neighborhood of \(k^2 = 1\). To do so, we rewrite (18) in the following form

\[
K_{\mu}(k, m) = \frac{1}{\Gamma(\pm) \sum_{n=0}^{2m}} \frac{(-1)^n 2^n \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + \frac{1}{2} \right)}{(2k^2)^{n+\frac{1}{2}} \Gamma(n + 1)}
\]
\[
\int_0^{\infty} e^{-2k^2 x^{\frac{1}{2}}} \sin^{\frac{1}{2} - \mu - \frac{1}{2}} \Phi \left( \frac{1}{2}, n + 1; x \right) dx
\]
We have therefore expressed $K_{\mu}(k,m)$ as a Laplace transform in which the coefficient in the first exponential, $(1-k^2)/k^2$, approaches zero as $k^2$ tends to 1. Hence we can now apply an Abelian theorem [11, p. 25] for Laplace transform to determine the behaviour of $K_{\mu}(k,m)$ in the neighbourhood of $k^2=1$. We know that the asymptotic expansion of the confluent hypergeometric function is [10, p. 271 or 12, p. 256]

$$\Phi(\alpha, \gamma; x) = \frac{\Gamma(\gamma) e^{\alpha x} x^{\alpha-\gamma}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\gamma-\alpha)_n (1-\alpha)_n}{n!} x^{-n}$$

(27)

as $x$ tends to infinity. Consequently an expansion formula for $K_{\mu}(k,m)$ valid in the neighbourhood of $k^2=1$ is

$$K_{\mu}(k,m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} (2m)!^r$$

(28)

provided that $(v-r)$ is neither zero nor a positive integer. We observe that

$$K_{\mu}(k,0) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}) \Gamma(n+\frac{1}{2}) \Gamma(\mu-n)}{\Gamma(\frac{1}{2}n! (1-k^2)^n-n^2 (2k^2)^{n+r+\frac{1}{2}})}$$

(29)

in the neighbourhood of $k^2=1$.

Let $p - r = \nu_n$, a positive integer. In this case, we decompose the infinite sum of (28) into two parts, $\sum_{n=0}^{\nu_n-1}$ and $\sum_{n=\nu_n}^{m}$. As the second sum presents an indeterminate form, we take the limit and use the formula [13, p. 13],

$$\psi(m-n)/\Gamma(m-n) \approx (-1)^{n-m+1} (n-m)!$$

(30)

$n > m$

to obtain an asymptotic expansion valid in the neighbourhood of $k^2=1$. 
If we set \( m = 0 \), \( \mu = j \), in (31) we get

\[
K_j(k,0) = \Omega_j(k) = \frac{-2j-1}{\Gamma(j+\frac{1}{2})} \sum_{n=0}^{j-1} \frac{\Gamma(n+\frac{1}{2}) \Gamma(j-n)}{\Gamma(j-n) \Gamma(n+\frac{1}{2})} \frac{k^2}{1-k^2} \frac{1}{n!} 2^{\frac{n+1}{2}} \frac{1}{2^{n-j}} \text{log} \left( \frac{1-k^2}{k^2} \right)
\]

5. RECURRENC FORMULAE

From the definitions of \( K_{\mu}(k,m) \) we can obtain a number of recurrence relations. For example, if we replace the numerator of (2) by \((1 - \sin^2\theta) \cos^{2m-2}\theta\) and then integrate by parts, we obtain

\[
K_{\mu}(k,m) = K_{\mu}(k,m-1) + \frac{(2m-2)}{k^2 (m-\frac{1}{2})} K_{\mu-1} (k, m - \frac{3}{2})
\]

If we multiply the numerator and denominator of (2) by \((1 - k^2 \cos\theta)\) and then decompose into two integrals, we get the following formula

\[
K_{\mu}(k,m) = K_{\mu+1}(k,m) - k^2 K_{\mu+1}(k,m+\frac{1}{2})
\]
From (33) and (35), we get

$$K_{\mu}(k,m) = K_{\mu}(k,m+1) + \frac{2m}{k^2\left(\frac{1}{2}-\mu\right)} K_{\mu-1}(k,m+\frac{1}{2}) 
- \frac{(1+2m)}{k^2\left(\frac{1}{2}-\mu\right)} K_{\mu-1}(k,m+\frac{3}{2})$$

From (33) and (35), we get

$$k^2\left(\frac{1}{2}-\mu\right) \left[K_{\mu}(k,m-1) - K_{\mu}(k,m+1)\right] 
= K_{\mu-1}(k,m-\frac{1}{2}) - (1+2m) K_{\mu-1}(k,m+\frac{1}{2}) 
+ 2(1-m) K_{\mu-1}(k,m-\frac{3}{2})$$

6. NUMERICAL COMPUTATION

In table 1, we compute the generalized elliptic-type integrals $K_{\mu}(k,m)$ on a VAX/VMS electronic computing machine, using equations (11) and (12). The results are given to eight significant digits.

In table 2, we compare some values of $K_{\mu}(k,m)$ from eqs. (11) and (12) ($m=0$, $\mu=j$, non negative integer) with the results given by (Epstein and Hubbell), Table 1).

In table 3, we compute the asymptotic approximations (24) and (28) of $K_{\mu}(k,m)$; $K_1$ and $K_2$ represent equations (24) and (28) respectively; the results are given in double precision.

In table 3, we observe a little difference in the entries of the two columns $K_1$ and $K_2$, and this is to be expected, due to the fact that the formula (24) is an approximate one, as only the first term of the hypergeometric series has been taken into consideration.

We propose to study some more properties and the numerical tabulation of $K_{\mu}(k,m)$ using the representation (19) and (31) in a subsequent paper.
### Table 1

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### REFERENCES


Resumo

Neste trabalho nós estudamos uma família de integrais de forma

\[ K_{\lambda}(k, m) = \int_0^\pi \frac{\cos^2 \theta d\theta}{(1 - k^2 \cos \theta)^{1+\frac{1}{2}}} \]

onde \( 0 < k < 1 \), \( \text{Re}(\mu) > -\frac{1}{2} \) and \( m \) é um inteiro não negativo. Nós obtemos uma expansão em série e estabelecemos sua relação com a função hipergeométrica de Gauss. Expansões assintóticas válidas em uma vizinhança de \( k=1 \) são fornecidas. Uma dessas fórmulas é obtida usando-se um teorema abeliano. Algumas relações de recorrência são estabelecidas. Resultados obtidos anteriormente por Epstein e Hubbell e por Weiss se seguem como casos particulares de nossas fórmulas. Alguns valores numéricos de \( K_{\lambda}(k, m) \) para valores selecionados dos parâmetros são tabelados, usando diferentes fórmulas.