The Quantum Field Theory Associated with the Infinite Lattice Two-Dimensional Ising Model

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Abstract We show that the Schwinger functions associated with the infinite lattice correlation functions of the periodic two-dimensional Ising model can be represented by a Feynmann-Kac (F-K) formula in a Fermion Fock space $\mathcal{H}$. The energy-momentum and field operators are expressed in terms of two sets of canonical Fermion free field operator-valued distributions acting in $\mathcal{H}$. These two sets are related by a proper linear canonical transformation (pct), i.e. there is a unitary operator $U$ which implements the transformation. Series representation for the Schwinger functions are obtained by substituting the spectral representations of the energy-momentum operators in the F-K formula. Below the critical temperature $P_+ = (I+U)/2$ are commuting orthogonal projections which reduce the algebra of observables and give an explicit decomposition of the periodic states into two, translationally invariant, pure states.

1. INTRODUCTION

Much attention has been devoted to the theoretical solution of the finite and infinite lattice two-dimensional Ising model of nearest neighbor interacting spins (see [1-7]). Also the time-continuum and scaling limits of this model have been studied (see [8-11]). In ref. 4 for finite volume and periodic boundary conditions an efficient method for handling the algebraic complexity is developed which employs two finite auxiliary sets of operators, $\{\xi_k, \xi_k^*\}$ and $\{\xi_\ell, \xi_\ell^*\}$ (the $\{k\}$ and $\{\ell\}$ are wavenumbers belonging to distinct sets) which satisfy anticommutation relations. All the eigenvalues and eigenfunctions of the transfer matrix are obtained explicitly. In ref. 7 a linear relation between the $\{\xi_k, \xi_k^*\}$ and $\{\xi_\ell, \xi_\ell^*\}$ operators is exploited to obtain a series representation for the n-point spin correlation functions, for finite and infinite volume.
Here we construct the infinite lattice quantum field theory of this model starting from the LMS solution (which is reviewed in section 2) preserving as much of the algebraic structure as possible. In section 3 for the infinite volume theory we introduce two sets of auxiliary free Fermion operator-valued distributions \( \{ \tilde{\xi}(k), \tilde{\xi}^*(k) \} \) and \( \{ \tilde{\xi}(k), \tilde{\xi}^*(k) \} \), \( k \in [-\pi, \pi] \) acting in the Fermionic Fock space \( H \). As in the finite volume case the \( \sim \) and \( \tilde{\sim} \) operators are not independent but are shown to be related by a proper linear canonical transformation (plct), \( J \), i.e., the transformation is implemented by a unitary operator \( U \) acting on \( H \) and have vacuum vectors \( \psi \) and \( \psi = U\tilde{\psi} \), respectively. Also energy-momentum and spin or field operators are defined. A Feynman-Kac formula is obtained for the infinite volume Schwinger functions in section 4 and an infinite series expansion results by inserting the spectral representations of the energy-momentum operators. Using the explicit form of \( \tilde{\psi} \) as given by the theory of plct (see Berezin) evaluation of the series is in principle reduced to an application of Wick's theorem. However, in ref. 18 a generalization of Wick's theorem is proved and used to evaluate the terms of the series; this same expansion has been obtained by Abraham using infinite systems of integral equations. The inverse of the plct, \( J \), is \( J \) which implies that \( U \) can be chosen to satisfy \( U^2 = I \) so that \( U^{-1} = U^* = U \). In section 5 we show that below the critical temperature the periodic state \( (\tilde{\psi}, \tilde{\psi}) \) admits a non-trivial decomposition \( (\tilde{\psi}, \tilde{\psi}) = 1/2 (\psi_+, \psi_+) + 1/2 (\psi_-, \psi_-) \) into the two translationally invariant states \( (\psi_+, \psi_+) \) where \( \psi_\pm = \sqrt{2} \) \( P_\pm \tilde{\psi} = (\tilde{\psi} \pm \tilde{\psi})/\sqrt{2} \) and \( P_\pm = (I \mp U)/2 \) are orthogonal projections satisfying \( P_+ P_- = 0 \). \( P_\pm \) reduce the algebra of observables. This decomposition has also been considered in \(^{15, 17}\)

A proof of the convergence of the correlation function for all temperatures is given in appendix A. The operator \( U \) and vacuum vector \( \tilde{\psi} \) are constructed explicitly in appendix B. In appendix C it is shown that the decomposition of the periodic state is non-trivial by showing that the magnetization is non-zero.

2. REVIEW OF THE LIEB-MATTIS-SCHULTZ SOLUTION OF THE TWO-DIMENSIONAL ISING MODEL
The partition function, $Z_{N,M}^{\ast}$, for the periodic Ising model with nearest neighbor spin interactions wrapped on a $N \times M$ torus is taken to be

$$Z_{N,M}^{\ast} = \sum_{\{\sigma\}} \exp \left\{ \sum_{m=1}^{M} \left( \sum_{n=1}^{N} K\sigma(n,m)\sigma(n+1,m) + K\sigma(n,n)\sigma(1,n) \right) \right\}$$

$$+ \sum_{n=1}^{N} \left( \sum_{m=1}^{M-1} K\sigma(n,m)\sigma(n,m+1) + K\sigma(n,m)\sigma(n,1) \right)$$

where $K = J T^{-1} > 0$ with $J > 0$, a constant, and $T$ the temperature. $\sigma(n,m)$ is the spin variable at the lattice point $n,m$ which takes values $\pm 1$ and $\sum$ is the sum over all $2^{NM}$ spin configurations.

Letting

$$\tau^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\tau^+\tau^-I, \quad \tau^\pm = \frac{1}{2}(\tau^x \pm i\tau^y)$$

we define

$$\tau^\hat{i}_m = 1 \otimes \ldots \otimes \tau^\hat{i} \otimes \ldots \otimes , \quad \hat{i} = x, y, z, +, -,$$

where $\tau^\hat{i}$ occurs in the $m$th factor from the left. The $\tau^\pm_m$ obey a mixed set of commutation, anti-commutation relations

$$[\tau^+_m, \tau^-_n] = 0, \ m \neq n ; \ \{\tau^+_m, \tau^-_m\} = 1; \quad (\tau^\pm_m)^2 = 0.$$

These operators act in the $2^M$ dimensional Hilbert space $H_M^0 = \bigotimes_{m=1}^{M} H_m$ where $H$ is the 2-dim. space generated by $|1\rangle$ and $|0\rangle$. In terms of $\tau^\pm_m$ the partition function can be written as the trace of an operator which we state as

Theorem 2.1 (see Huang\(^{\ast}\)): $Z_{N,M}^{\ast} = \text{Tr}(V_1^{1/2} V_2^{1/2} V_1^{N}) \equiv \text{Tr} V^{N}$

where

$$V_1 = (2\sinh 2K)^{M/2} \exp \left[ -2K^* \sum_{m=1}^{M} (\tau^+_m \tau^-_m - 1/2) \right]$$

and
The critical temperature, \( T_c \), is defined by \( K^* = K \) or equivalently \( \sinh \left( \frac{2J}{T_c} \right) = 1 \).

Let \( B_M = V_1^{1/2} V_1 V_2^{1/2} \) which is self-adjoint. By suitable transformations \( B_M \) can be written as

\[
B_M = (2 \sinh 2K)^{M/2} e^{-H_M}.
\]

\( H_M \) is identified as the finite volume Hamiltonian,

The first transformation is a Jordan Wigner transformation. Let

\[
c_m = \exp \left[ \pi i \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^- \right] \tau_1^-, \quad 1 < m \leq M; \quad \tau_1 = \tau_1^-,
\]

the inverse transformation is

\[
\tau_m^x = \left[ \exp \left( \pi i \sum_{j=1}^{m-1} \sigma_j^x \sigma_j^x \right) \right] \sigma_m^x, \quad \tau_m^+ = \left[ \exp \left( \pi i \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^- \right) \right] \sigma_m^x,
\]

thus

\[
\tau_m^x = \left[ \exp \left( \pi i \sum_{j=1}^{m-1} \sigma_j^x \sigma_j^x \right) \right] \sigma_m^x, \quad \sigma_m^x = \sigma_m^x + \sigma_m^x.
\]

The advantage of this transformation is that the \( c_m \) obey anti-commutation relations such as \( \{ c_m, c_m^* \} = \delta_{mm'} \), \( \{ c_m, c_m' \} = \{ c_m^*, c_m^*' \} = 0 \). The vector

\[
\Omega = \left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) \in H_M
\]

with M factors has the property \( c_m R = 0 \) for all \( m \). A basis for \( H_M \) is

\[
\Omega; \quad \{ c_m^* \Omega \}; \quad \{ c_{m_1}^* c_{m_2}^* \Omega \}, \quad m_1 < m_2; \quad \cdots \{ c_1^* \cdots c_M^* \Omega \}.
\]

\( H_{e(0)} \) denotes the subspace generated by the vectors with an even (odd) number of \( c_m \)'s, the subspace corresponding to the eigenvalue 1 (-1) of the operator
N = \exp(\pi i \sum_{m=1}^{M} \sigma_m^* \sigma_m) = \exp(\pi i \sum_{m=1}^{M} \gamma_m^+ \gamma_m^-) = \frac{1}{M} \left( -\varepsilon_0^B \right).

Secondly, a linear canonical transformation is defined by

\[ \sigma_m = M^{-1/2} \alpha \sum_{q} e^{i \omega q} \eta_q, \quad \alpha = e^{-i \pi/4}, \]

\[ q \in S^+ \text{ or } S^-, \text{ but not both, where } S^\pm \text{ are the } M \text{ element sets} \]

\[ S^+ = \{ q \in \mathbb{Z} = \pm \pi/M, \pm 3\pi/M, \ldots \pm (M-1)\pi/M \} \]

\[ S^- = \{ q \in \mathbb{Z} = 0, \pm 2\pi/M, \ldots \pm (M-2)\pi/M, \pi \} \]

and \( M \) is assumed to be even. Throughout this section the letter \( \xi(k) \) will refer to elements of \( S^+ \) (or \( S^- \)).

Finally, we make the linear canonical Valatin-Bogoliubov transformation

\[ E_q = \cos \phi \eta_q + \sin \phi \eta_q^* \]

where

\[ \tan \phi = C / (e^{E_q} - A), \quad q \in S^+ U S^-/\{0, \pi\}, \]

or \( \tan 2\phi = 2C / (B - A) \) and we require \( \phi_q = -\phi_q^*, \phi_q \geq 0 \) for \( q > 0 \).

For \( q = 0 \) we set \( \phi_0 = 0 \) for \( T > T_c \); for \( T < T_c \), \( \phi_0 = \pi/2, \xi_0 = \eta_0^* \), and for \( q = \pi, \phi_\pi = 0 \). The choice \( \phi_0 = \pi/2 \) for \( T < T_c \) differs from that of LMS and is made to simplify the expression for \( H_4 \) in Thm. 1.3, i.e. \( \varepsilon_0 \geq 0 \).

\[ A_4 = e^{-2K^*} (\cosh K + \sinh K \cos q)^2 + e^{2K^*} (\sinh K \sin q)^2, \]

\[ B_q = -2K^* (\sinh K \sin q)^2 + e^{2K^*} (\cosh K - \sinh K \cos q)^2, \]

\[ C_q = (2\sinh K \sin q) (\cosh 2K^* \cosh K - \sinh 2K^* \sinh K \cos q), \]

\[ \cosh E_4 = \cosh 2K^* \cosh 2K - \sinh 2K^* \sinh 2K \cos q, \quad E_q \geq 0. \]
It is to be noted that for $T < T_c$, $\phi_q$ is discontinuous at $q = 0$; for $T > T_c$, $\phi_q$ is continuous. $\Theta(e^{i\theta}) = e^{-2i\theta}$ admits an analytic continuation in the variable $z = e^{\theta}$ which is

$$\Theta(z) = -z \left[ \frac{(1-x_1^{-1}z)(1-x_2z)}{(1-x_1^{-1}z^{-1})(1-x_2z)} \right]^{1/2}$$

for $T < T_c$,

$$\left[ \frac{(1-x_1^{-1}z)(1-x_2z^{-1})}{(1-x_1^{-1}z^{-1})(1-x_2z^{-1})} \right]^{1/2}$$

for $T > T_c$.

where $x_1 = \sinh K^* \sinh K > 1$, $x_2 = (\sinh K/\sinh K^*)^{-1}$ for $T < T_c$ and the phase angle in each factor of $\Theta(z)$ is taken to be in $(0, \pi)$. The winding number of $\Theta(z)$ is $\pm 1(0)$ for $T < T_c$ ($T > T_c$). Even though $\phi$ does not have a simple expression in terms of $\xi_q$, $c_m + \alpha_m$ does, and we find

$$c_m + \alpha_m = (\alpha \sum_q e^{i\xi_m} e^{i\phi_q} \xi_q + \tilde{\alpha} \sum_q e^{-i\xi_m} e^{-i\phi_q} \xi_q) M^{-1/2}$$

where $q \in S^+$ or $S^-$, but not both.

In terms of the $\xi_q$ operators we have:

Thm. 2.3.

$$H_M = H_M^+ + H_M^-,$$

$$H_M^+ = \sum_{k \in S^+} \xi_k (\xi^*_k \xi_k - 1/2)$$

$$H_M^- = \sum_{k \in S^-} \xi_k (\xi^*_k \xi_k - 1/2)$$

The sets $\{\xi_k\}_{k \in S^+}$ and $\{\xi_k\}_{k \in S^-}$ obey anti-commutation relations, however they are not independent; from

$$c_m = M^{-1/2} \sum_{k \in S^+} e^{i\xi_m} \eta_k$$

$$= M^{-1/2} \sum_{k \in S^-} e^{i\xi_m} \eta_k$$

$$= M^{-1/2} \sum_{k \in S} e^{i\xi_m} \eta_k$$
upon multiplying by $\alpha e^{-ik'm}$, summing over $M$ and replacing $k'$ with $k$ we obtain

$$\eta_k = \alpha \frac{1}{M} \sum_m e^{-ik'm} \sum_R e^{ikm} \eta_k,$$

which upon substituting in

$$\xi_k = \cos \phi_k \eta_k + \sin \phi_k \eta^*_k$$

gives

$$\xi_k = t_{1k} \xi_k + t_{2k} \xi_k = \frac{1}{M} \sum_{\ell \in S^+} \left\{ \frac{2e^{-i(k-\ell)}}{1-e^{-i(k-\ell)}} \right\} \cos (\phi_k - \phi_\ell) \xi_k + \sin (\phi_k - \phi_\ell) \xi^*_k \right\},$$

a linear canonical transformation between $\{\xi_R \}_{R \in S^+}$ and $\{\xi_k \}_{k \in S^-}$. If $T > T_c$, the same relation holds with $k$ and $\ell$ interchanged. But for $T < T_c$:

$$\xi_k = \frac{1}{M} \sum_{\ell \in S^-} \left\{ \frac{2e^{-i(k-\ell)}}{1-e^{-i(k-\ell)}} \right\} \cos (\phi_k - \phi_\ell) \xi_k + \sin (\phi_k - \phi_\ell) \xi^*_k \right\},$$

The vacuum vectors $\psi_{S^+(S^-)}$ which satisfy $\sum q \psi_{S^+(S^-)} = 0$ for all $q \in S^+(S^-)$ are given by

$$\psi_{S^+} = \prod_{\ell \in S^+} \left( \cos \phi_\ell + \sin \phi_\ell \eta^*_\ell \eta_\ell \right)$$

$$\psi_{S^-} = \prod_{\ell \in S^-} \eta^*_\ell \cos \phi_\ell + \sin \phi_\ell \eta^*_\ell \eta_\ell \right)$$

for $T < T_c$.

so that $\psi_{S^+} \in H_e$ and $\psi_{S^-} \in H_0(e)$ for $T < T_c$ ($T > T_c$).
The lowest eigenvalue of $H_M^+$ is $-\frac{1}{2} \sum_k E_k$ and of $H_M^-$ is $-\frac{1}{2} \sum_k E_k'$.

By the unitary transformation,

$$U = \prod_{m=1}^{M} U_m, \quad U_m = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we have

$$U^T x_m U^{-1} = -\tau^x_m, \quad U^T \tau^x_m U^{-1} = \tau^x_m, \quad U^T e_m U^{-1} = \tau^e_m$$

and $U e^{-H_M} U^{-1}$ has positive matrix elements. By the Perron-Frobenius theorem the eigenspace associated with the largest eigenvalue of $U e^{-H_M} U^{-1}$ is one-dimensional and the corresponding eigenvector, $\phi'$, can be chosen to have strictly positive components which implies $\phi' e U e^{-H} e = e' e$.

Thus, the eigenvector associated with the lowest eigenvalue of $H_M$ is $\psi_S \in H_e$.

In terms of the $\xi$ operators, the momentum operator, $P_M$, is given by

$$P_M = P_M^+ \oplus P_M^-,$$

Thm. 2.4.\footnote{19}

$$P_M^+ = \sum_k \xi_k \xi^*_k \xi^*_k,$$

$$P_M^- = \sum_k \xi_k \xi^*_k \xi^*_k$$

and the equations

$$\epsilon m e^{-\epsilon P_M} x_m = \epsilon m e^{-1}, \quad 2 \leq m \leq M$$

$$\epsilon m e^{-\epsilon P_M} x_1 e^{-\epsilon P_M} = \epsilon_m$$

are satisfied.

The finite volume correlation functions are given by

Thm. 2.5. If $1 \leq n_1, n_2, \ldots, n_k \leq N$, then
\[
\langle \sigma(n_1, m_1) ... \sigma(n_k, m_k) \rangle_{N,M} = \frac{\text{Tr}(e^{-H_M n_1} x^{-H_M(n_2-n_1)} x ... x^{-H_M(N-n_k)})}{\text{Tr}(e^{-H_M N})}
\]

Let \( \hat{H}_M = H_M - \inf \text{spec } H_M \) and \( \widehat{H}_M(\cdot) = \hat{H}_M \uparrow H(e(0)) \). Letting \( N \rightarrow \infty \) we obtain the finite volume Feynman-Kac formula as

**Corollary 2.5.1.** If \( 1 = n_1 \leq ... \leq n_k \), then

\[
S_{Mk} \equiv \langle \sigma(n_1, m_1) ... \sigma(n_k, m_k) \rangle_M = \langle \psi_{\mathcal{S}^+}, \tau_{m_1} e^{-\hat{H}_M(n_2-n_1)} \tau_{m_2} ... \tau_{m_k} \psi_{\mathcal{S}^+} \rangle
\]

Remark: By making a 1-1 correspondence between \( \mathcal{S}^+ \) and \( \mathcal{S}^- \) Berezin's theory of plate \( \xi^2 \) gives a unitary \( U_M \) such that \( \xi K = U_M^* \xi U_M^{-1} \). Knowing the null space of \( \tau_1 \), \( U_M \) and \( U_M^{-1} \) are given explicitly in terms of \( \{ \xi_\ell \} \) so that the energy-momentum factors with \( \hat{H}_M, \hat{P}_M \) can be expressed in terms of only the one set of fermions \( \{ \xi_\ell \} \).

A series representation for \( S_{Mk} \) is obtained by inserting the spectral representation of \( H \) and \( P \). For \( T > T_c \),

\[
\xi^*_{K_1} \psi_{\mathcal{S}^-}, \xi^*_{K_2} \xi^*_{K_3} \psi_{\mathcal{S}^-}, ... , \xi^*_{K_c} ... \xi^*_{K_{M-1}} \psi_{\mathcal{S}^-}
\]

generate \( H_0 \) and \( H_e \), respectively. Thus we have

**Corollary 2.5.2.**
where $\gamma_{\alpha j}(\gamma_{\beta j})$ is the eigenvalue of $\tilde{H}_M + iP_M^-(\tilde{H}_M + iP_M^+)$ corresponding to the eigenvector $\chi_{\alpha j}(x_{\beta j})$ with

$$\chi_{\alpha j} = (n!)^{-1/2} \xi_{k_1}^* \ldots \xi_{k_n}^* \psi_{S^-} \quad \alpha_j = n \text{ odd}$$

$$\chi_{\beta j} = (n!)^{-1/2} \xi_{l_1}^* \ldots \xi_{l_n}^* \psi_{S^+} \quad \beta_j = n \text{ even}.$$

The $\alpha_j(\beta_j)$ are multi-indices taking odd (even) values. A similar expansion holds for $T < T_c$ where

$$\psi_{S^-}, \xi_{k_1}^* \xi_{k_2}^* \psi_{S^-}, \ldots, \xi_{k_1}^* \xi_{k_2}^* \ldots \xi_{k_M}^* \psi_{S^-} \quad \text{and}$$

$$\psi_{S^+}, \xi_{l_1}^* \xi_{l_2}^* \psi_{S^+}, \ldots, \xi_{l_1}^* \xi_{l_2}^* \ldots \xi_{l_M}^* \psi_{S^+}$$

generate $H_0$ and $H_e$, respectively.

Remarks: 1. As noted by Abraham\textsuperscript{7} by using the expression for $\tau_1^X$ in terms of the $\xi_q$'s and the linear relations $\xi_q = s_{1k} \xi_k + s_{2k} \xi_k^*$, Corollary 2.5.2 reduces the determination of the correlation functions to the determination of a generic matrix element

$$(\psi_{S^-}, \xi_{k_1} \ldots \xi_{k_n} \psi_{S^+})$$

or

$$(\psi_{S^-}, \xi_{l_1}^* \ldots \xi_{l_n}^* \psi_{S^+})$$

since $\xi_{k} \psi_{S^+} = 0$ and $\xi_{k}^* \psi_{S^-} = 0$.

2. By Berezin's\textsuperscript{12} theory of plct if we know the null space of $t_1^X$ or $s_1^X$ then we have explicit formulas for $\psi_{S^-}$ and terms of $\{ \xi_q \}$ and $\psi_{S^+}$ and the above matrix elements can be obtained by applying Wick's theorem.
This can also be done in the infinite lattice case but in a generalization of Wick's theorem is proved and used to obtain the matrix elements.

3. CONSTRUCTION OF INFINITE VOLUME OPERATORS AND THEIR PROPERTIES

In this section we define infinite volume analogs of the operators $\xi_k$, $\xi^*_k$, $\eta_k$, $\eta^*_k$, $H_M^\pm$, $P_M^\pm$, $c_m$, $q_m^\infty$ of section 2 and study their properties. The theory of proper (improper) linear canonical transformations plct (ilct) as developed in Berezin enters in an essential way. For relevant definitions, results and proofs we refer the reader to 12. In section 4 these operators will be used to define the infinite volume correlation functions as a Feynmann-Kac formula.

Let $H$ be the anti-symmetric Fock space over $L^2(-\pi,\pi)\mathbb{R}$ and let $\hat{\xi}(k)$, $\hat{\xi}^*(k)$ be the usual Fock annihilation and creation operator-valued distributions satisfying the canonical anti-commutation relations

$$\{\hat{\xi}(k), \hat{\xi}(k')\} = \{\hat{\xi}^*(k), \hat{\xi}^*(k')\} = 0, \quad \{\hat{\xi}(k), \hat{\xi}^*(k')\} = \delta(k-k')$$

and let $\hat{\psi}$ denote the vacuum vector, i.e. $\hat{\xi}(f)\hat{\psi} = 0$ for all $f \in \mathcal{F}$.

We define the Hilbert transform, $H$, as the $L^2(-\pi,\pi)$ closure of $H_C$ where

$$f(k) = \pi^{-1} \int_{-\pi}^\pi (1 - e^{-i(k-q)})^{-1} f(q) dq,$$

for $f \in C[-\pi,\pi]$. $H$ satisfies $|H| = 1$ and $H^2 = I$. For any bounded operator $A$ on $L^2(-\pi,\pi)$ define $\tilde{A}$ by $(\tilde{A}f)(k) = \overline{(A\overline{f})(k)}$ and $A^* = \overline{A^*}$. Thus $H^* = H$.

Also define the operator $V$ by $(Vf)(k) = f(-k)$. Thus $V^* = V'$. The analogs of $\xi_k$ and $\xi^*_k$ are defined by

$$\tilde{\xi}(f) = \xi(T_1f) + \xi^*(T_2f), \quad (2.1a)$$

$$\tilde{\xi}^*(f) = \xi(T_2f) + \xi^*(\overline{T}_1f), \quad (2.1b)$$
\( \mathbf{f} \in L^2; T_1, T_2 \) are the \( L \) closures of \( T_{1\alpha}, T_{2\alpha} \) where

\[
(T_{1\alpha}\mathbf{f})(k) = \pi^{-1} \int_{-\pi}^{\pi} (1-e^{i(k-q)})^{-1} \cos(\phi + \phi_k) f(q) dq
\]

(2.2)

\[
(T_{2\alpha}\mathbf{f})(k) = \pi^{-1} \int_{-\pi}^{\pi} (1-e^{i(k+q)})^{-1} \sin(\phi + \phi_k) f(q) dq
\]

(2.3)

\( \mathbf{f} \in \mathcal{C}[-\pi, \pi] \), and \( \mathcal{P} \) denotes the principal value. Some of the properties of \( T_1 \) and \( T_2 \) are given by

**Thm. 3.1,**

i) \( T_1 = \cos \phi \mathcal{H} \cos \phi + \sin \phi \mathcal{H} \sin \phi = 2^{-1}(e^{i\phi} \mathcal{H} e^{-i\phi} + e^{-i\phi} \mathcal{H} e^{i\phi}) \)

ii) \( T_2 = \cos \phi \mathcal{VH} \sin \phi + \sin \phi \mathcal{VH} \cos \phi = (2i)^{-1}(e^{i\phi} \mathcal{VH} e^{-i\phi} - e^{-i\phi} \mathcal{VH} e^{i\phi}) \)

iii) \( T_1 \neq \mathcal{T}_1, \ T_1 = T_1^*, \ T_1 \neq \mathcal{T}_1 \)

iv) \( T_2 \neq \mathcal{T}_2, \ T_2 \neq \mathcal{T}_2 \)

v) \( T_1^* T_2 + T_2^* T_1 = 0 \)

vi) \( T_1 \mathcal{T}_2 + T_2 \mathcal{T}_1 = \mathcal{I} \)

vii) \( \dim \ker T_1 = 1(0) \) for \( T < T_\alpha \) \( (T > T_\alpha) \)

= index \( \mathcal{O}(z) \).

**Prf. of Thm 3.1:**

i) - iv) follow from the definitions of \( T_1 \) and \( T_2 \).

v) is proved by direct calculation. We have

\[
T_1^* T_2 + T_2^* T_1 = (\cos \phi \mathcal{VH} \cos \phi + \sin \phi \mathcal{VH} \sin \phi) + \\
(\cos \phi \mathcal{VH} \sin \phi + \sin \phi \mathcal{VH} \cos \phi) + \\
(\cos \phi \mathcal{VH} \sin \phi + \sin \phi \mathcal{VH} \cos \phi) + \\
(\cos \phi \mathcal{VH} \cos \phi + \sin \phi \mathcal{VH} \sin \phi) = \\
= \cos \phi \mathcal{VH} (\cos^2 \phi \mathcal{V} + \sin^2 \phi \mathcal{H}) \sin \phi + \\
\sin \phi \mathcal{VH} (\sin^2 \phi \mathcal{V} + \cos^2 \phi \mathcal{H}) \cos \phi + \\
\cos \phi \mathcal{VH} (\cos^2 \sin \phi \mathcal{V} + \sin \phi \cos \phi \mathcal{H}) \cos \phi + \\
\sin \phi \mathcal{VH} (\sin^2 \cos \phi \mathcal{V} + \cos \phi \sin \phi \mathcal{H}) \sin \phi
\]

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The last two terms vanish since \( \sin \phi \cos \phi \) is odd. Since \( \sin^2 \phi \) and \( \cos^2 \phi \) are even and \( H^2 = I \) the right side becomes

\[
\cos \phi V \sin \phi + \sin \phi V \cos \phi = V(\cos \phi \sin \phi - \sin \phi \cos \phi) = 0.
\]

vi) is proved in a similar way. vii) is proved by observing \( T'_1 f = 0 \) iff \( H \Theta g + \Theta H g = 0 \) where \( g = e^{i\phi} f \) and \( \Theta = e^{-2i\phi} \). Abraham has shown that the kernel of \( H \Theta + \Theta H \) obeys the statement of vii).

Concerning the operators \( \tilde{\xi}(f), \tilde{\xi}^*(f) \) defined by eq. (2.1a,b) we have Thm. 3.2,

i) \( \tilde{\xi}(f), \tilde{\xi}^*(f) \) satisfy CACR.

ii) eqs. (2.1a,b) hold with \( \sim \) and \( \sim \) interchanged.

iii) the transformation (2.1a,b) is a plct, i.e., there exists a unique (up to a constant \( \lambda, |\lambda| = 1 \)) unitary \( U \) such that for all \( f \in L \)

\[
\tilde{\xi}(f) = U \tilde{\xi}(f) U^{-1}, \quad \tilde{\xi}^*(f) = U \tilde{\xi}^*(f) U^{-1}
\]

and \( \tilde{\psi} = U\tilde{\psi} \) is the vacuum vector for \( \tilde{\xi}(f) \), i.e. \( \tilde{\xi}(f) \tilde{\psi} = 0 \) for all \( f \in L \).

iv) the \( U \) of iii) satisfies \( U^2 = aI, |a| = 1 \). \( U \) and \( \tilde{\psi} \) can be redefined so that \( U = U^* \) and \( \tilde{\psi} = U\tilde{\psi} \).

Remarks:

1. In the sequel we take \( U = U^* \); this choice plays an important role in the 'one-Fermion' Schwinger functions (see Theorem 4.2) and in the decomposition of the periodic state in section 5.

2. Formulas for \( U \) and \( \tilde{\psi} \) are given in Appendix B in terms of \( \tilde{\xi}(k), \tilde{\xi}^*(k) \) and \( \tilde{\psi} \).

Prf. of Thm. 3.2: i) Follows from v) and vi) of Thm. 3.1. ii). We see that

\[
\tilde{\xi}(T_1 f) = \tilde{\xi}(T_1^2 f) + \tilde{\xi}^*(T_2 T_1 f),
\]

\[
\tilde{\xi}^*(T_2 f) = \tilde{\xi}(\tilde{T}_2 T_2 f) + \tilde{\xi}^*(\tilde{T}_1 T_2 f).
\]
From Thm. 3.1 vi), \( T_1^* T_1 + T_2^* T_2 = \mathbb{1} \) or \( T_1^2 + T_2^2 = \mathbb{1} \) (since \( T_1^* = T_1 \) and \( T_2^* = T_2 \)). Thus \( T_1^2 + T_2^2 = \mathbb{1} \). From Thm. 3.1 v), \( T_2^* T_1 + \mathbb{1} T_2^* T_2 = 0 \) or \( T_2^* T_1 + T_1^* T_2 = 0 \). Thus
\[
\tilde{\xi}(T_1 f) + \tilde{\xi}^*(T_2 f) = \tilde{\xi}((T_1^2 + T_2^2) f) + \tilde{\xi}^*((T_2 T_1 + T_1 T_2) f) = \tilde{\xi}(f).
\]

Similarly one shows \( \tilde{\xi}^*(f) = \tilde{\xi}(\overline{T_2} f) + \tilde{\xi}^*(\overline{T_1} f) \).

iii) From\(^{12}\), i) and ii) it is sufficient to show that \( T_1 \) has property A and that \( T_2 \) is Hilbert-Schmidt. As \( T_1 \) is self-adjoint it has property A. \( T_2(k,q) \), the kernel of \( T_2 \), is given by
\[
T_2(k,q) = \pi^{-1} \sin(\phi + q\kappa) (1 - e^{-i(k+q)})^{-1}
\]
Since \( T_2(k,q) \) is bounded and piecewise continuous on \( [-\pi, \pi] \times [-\pi, \pi] \) it is in \( L^2((-\pi, \pi) \times (-\pi, \pi)) \), thus it is Hilbert-Schmidt.

iv) By ii) and iii):
\[
U \tilde{\xi}(f) U^{-1} = U(\tilde{\xi}(T_1 f) + \tilde{\xi}^*(T_2 f)) U^{-1} =
\]
\[
= \tilde{\xi}(T_1 f) + \tilde{\xi}^*(T_2 f) = \tilde{\xi}(f) = U^{-1} \tilde{\xi}(f) U \]
or \( U^2 \tilde{\xi}(f) = \tilde{\xi}(f)^2 U \) for all \( f \in L \). Similarly \( U^2 \tilde{\xi}^*(f) = \tilde{\xi}^*(f) U^2 \). Thus \( U^2 \) commutes with all \( \tilde{\xi}(f) \), \( \tilde{\xi}^*(f) \) which (see ref.\(^{12}\)) implies \( U^2 = \lambda I \), \( |\lambda| = 1 \).

This result also follows abstractly from the following group property; associate the matrix
\[
J = \begin{pmatrix}
T_1^* & T_2^*

T_2 & T_1
\end{pmatrix}
\]
with the transformation of eqs. (2,1a,b). The lct form a group (successive transformations corresponding to matrix multiplication of the associated matrices) and the inverse transformation has the matrix
\[
J^{-1} = \begin{pmatrix}
(T_1^*)^* & (T_2^*)

(T_2)^* & (T_1^*)^*
\end{pmatrix}
\]
By Thm. 3.1 iii) and iv) \( T_1^\dagger = T_1 \) and \( T_2^\dagger = T_2 \) so that \( J^{-1} = J \) or \( JJ = I \).

As the unitaries which implement the transformations form a ray representation of the lct group we have \( U^2 = A \mathbb{1} \), \( |\lambda| = 1 \).
It is always possible to redefine $U$ and $\tilde{\psi} = U\tilde{\psi}$ such that $U^2 = I$. Indeed, for the new $U$ take the operator $4 = \mu U$ with $\mu^2 = \lambda^{-1}(|\mu|=1)$ and for the new $\tilde{\psi}$ the vector $\tilde{\psi}_n = \mu \tilde{\psi}_n$. Thus $U_n^2 = I$, $V_n = U_n^*$ and $\tilde{\psi}_n = V_n \tilde{\psi}_n$.

Let $\hat{\mathcal{H}}_e(\sigma)$ be the subspace of $\mathcal{H}$ with an even (odd) number of $\xi$-particles. More precisely $\hat{\mathcal{H}}_e(\sigma)$ is the eigenspace of $\exp(\pi \xi N)$ associated with the eigenvalue $+1(-1)$ where

$$\hat{\mathcal{N}} = \int_{-\pi}^{\pi} \hat{\xi}^*(k) \hat{\xi}(k) dk.$$

Similarly define $\hat{\mathcal{H}}_o(\sigma)$. We have

Thm. 3.3.

$$\hat{\mathcal{H}}_e = \hat{\mathcal{H}}_o, \quad \hat{\mathcal{H}}_o = \hat{\mathcal{H}}_e$$

for $T < T_c$ and

$$\hat{\mathcal{H}}_e = \hat{\mathcal{H}}_e, \quad \hat{\mathcal{H}}_o = \hat{\mathcal{H}}_o$$

for $T > T_c$.

Prf. of Thm. 3.3. Follows from the form of $\tilde{\psi}$ given in Appendix B.

In analogy with the finite lattice case we define infinite volume energy and momentum operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$, respectively, by

$$\hat{\mathcal{H}} \upharpoonright \hat{\mathcal{H}}_e = \int \varepsilon(k) \hat{\xi}^*(k) \hat{\xi}(k) dk \upharpoonright \hat{\mathcal{H}}_e,$$

$$\hat{\mathcal{H}} \upharpoonright \hat{\mathcal{H}}_o = \int \varepsilon(k) \hat{\xi}^*(k) \hat{\xi}(k) dq \upharpoonright \hat{\mathcal{H}}_o,$$

$$\hat{\mathcal{P}} \upharpoonright \hat{\mathcal{H}}_e = \int k \hat{\xi}^*(k) \hat{\xi}(k) dk \upharpoonright \hat{\mathcal{H}}_e,$$

$$\hat{\mathcal{P}} \upharpoonright \hat{\mathcal{H}}_o = \int q \hat{\xi}^*(q) \hat{\xi}(q) dq \upharpoonright \hat{\mathcal{H}}_o,$$

for $T < T_c$ and for $T > T_c$.

$$\hat{\mathcal{H}} \upharpoonright \hat{\mathcal{H}}_e = \int \varepsilon(k) \hat{\xi}^*(k) \hat{\xi}(k) d\tilde{k} \upharpoonright \hat{\mathcal{H}}_e,$$

$$\hat{\mathcal{H}} \upharpoonright \hat{\mathcal{H}}_o = \int \varepsilon(q) \hat{\xi}^*(q) \hat{\xi}(q) dq \upharpoonright \hat{\mathcal{H}}_o,$$

$$\hat{\mathcal{P}} \upharpoonright \hat{\mathcal{H}}_e = \int k \hat{\xi}^*(k) \hat{\xi}(k) dk \upharpoonright \hat{\mathcal{H}}_e,$$

$$\hat{\mathcal{P}} \upharpoonright \hat{\mathcal{H}}_o = \int q \hat{\xi}^*(q) \hat{\xi}(q) dq \upharpoonright \hat{\mathcal{H}}_o.$$

It is interesting to note
Thm. 3.4. Let
\[ \tilde{\eta}(q) = \cos\phi_q \xi(q) - \sin\phi_q \xi^*(-q), \]
\[ \tilde{\eta}(q) = \cos\phi_q \xi(q) - \sin\phi_q \xi^*(-q). \]

These transformations are improper linear canonical transformations, i.e. there does not exist a unitary \( U \) which implements the transformation. Furthermore \( \tilde{\eta}(f) = \tilde{\eta}(Hf) \).

Prf. of Thm 3.4: Write \( \tilde{\eta}(f) = \xi(f_1) + \xi^*(f_2) \) for \( f \in L^2 \) and
\[ f_1(k) \equiv (S_1f)(k) = \cos\phi_k f(k) \]
\[ f_2(k) \equiv (S_2f)(k) = \sin\phi_k f(-k). \]

By direct calculation we find
\[ S_1^*S_2 + S_2^*S_1 = 0 \]
\[ S_1^*S_1 + S_2^*S_2 = I \]
and as in the proof of Thm. 3.3 the lct is invertible,
\[ S(f) = \tilde{\eta}(S_1f) - \tilde{\eta}^*(S_2f), \]
and \((\tilde{\eta}(k), \tilde{\eta}^*(k))\) satisfy canonical anti-commutation relations. The transformation is improper, i.e. not proper, since \( S_2 \) is not a Hilbert-Schmidt operator.
\[ \tilde{\eta}(f) = \tilde{\eta}(Hf) \] follows from the explicit evaluation of
\[ \tilde{\eta}(f) = \xi(S_1f) + \xi^*(S_2f) \]
\[ = \xi((T_1S_1 + T_2S_2)f) + \xi^*((T_2S_1 + T_1S_2)f) \]
\[ = \tilde{\eta}((S_1T_1S_1 + S_1T_2S_2 - S_2T_2S_1 - S_2T_1S_2)f) + \]
\[ \tilde{\eta}^*((S_1T_2S_1 + S_1T_1S_2 - S_2T_1S_1 - S_2T_2S_2)f). \]

We now define
\[ \mathcal{C}_m = \frac{\alpha}{2\pi} \int e^{i\xi(m-1)} \tilde{\eta}(q) dq, \quad m \geq 1 \] and
\[ \tilde{c}_m = \frac{\alpha}{2\pi} \int e^{iq(m-1)} \tilde{n}(q) dq, \quad m \geq 1, \quad \alpha = e^{-i\pi/4}. \]

We have

**Thm. 3.5.** \( \tilde{c}_m = \tilde{c}_m, \quad m \geq 1. \)

**Prf. of Thm. 3.5:** Follows from \( H e^{iqm} = (\text{sgn } m)e^{iqm} \). Since there is no distinction between \( \tilde{c}_m \) and \( \tilde{c}_m \), we denote these operators by \( \tilde{c}_m \). We define the field operators, \( \tilde{c}_m \), by

\[ \tilde{c}_m = \tilde{c}_m = \frac{1}{2\pi} \int \left[ a e^{-i\tilde{q}(q)} \tilde{\xi}(q) + \tilde{\alpha} e^{-i\tilde{q}(q)} \tilde{\xi}^*(q) \right] dq \]

and

\[ \tilde{c}_m = e^{-i\tilde{p}_m} \tilde{c}_m e^{i\tilde{p}_m} \]

where \( \tilde{c}_m = \tilde{c}_m + \tilde{c}_m^* \).

We believe, but have been unable to prove that linear combinations of vectors of the forms

\[ \left( \prod_{i=1}^{N} e^{-H|n_i|} e^{i\tilde{p}_m} \tilde{c}_m \right)^* \tilde{\psi} \]

are dense in \( H \). Assuming this result, we have

**Thm. 3.6:** \( [\tilde{c}_m, \tilde{c}_n^*] = 0. \)

**Proof:** It is sufficient to show

\[ \left( \prod_{i=1}^{N} e^{-H|n_i|} e^{i\tilde{p}_m} \tilde{c}_m \right)^* \tilde{\psi}, \quad \tilde{c}_m \tilde{c}_n \left( \prod_{j=1}^{N'} e^{-H|n_j|} e^{i\tilde{p}_m} \tilde{c}_m \right)^* \tilde{\psi} \]

is symmetric in \( m, n \). But this follows from the Feynmann-Kac formula (whose proof will be given in sec. 4) which relates the above inner-product to an \( N + N' + 2 \) - point correlation function of the periodic Ising model.

**4. INFINITE VOLUME SCHWINGER FUNCTIONS AND THEIR REPRESENTATION**

In terms of the vectors and operators introduced in section 3, define the \( k \)-point Schwinger function, \( S_k \), as
where \( n_1 \leq n_2 \ldots \leq n_k \). We have

Thm. 4.1, \( S_k \) of eq. (4.1) is indeed the \( k \)-point correlation function of the periodic Ising model, i.e.

\[
S_k = \lim_{M \to \infty} (S_{Mk} = \langle \sigma(n_1,m_1) \ldots \sigma(n_k,m_k) \rangle_M) = \langle \sigma(n_1,m_1) \ldots \sigma(n_k,m_k) \rangle.
\]

We defer the proof of Thm. 4.1 to the end of this section.

From the definition of \( H \) and \( P \) in section 3 we see that the \( \sim \) or \( \bar{\sim} \) operators occur in \( H \) or \( P \) depending on the parity of the vector on which it acts. However, \( S_k \) of eq. (4.1) can be expressed in terms of the \( \sim \) operators only as seen in

Thm. 4.2. Let

\[
\mathcal{R}(P) = \int_{-\pi}^{\pi} \epsilon(k) \xi(k) \bar{\xi}(k) \, dk
\]

and \( U \) being expressed in terms of the \( \sim \) operators. Then \( \sigma_0 = U^{-1} \sigma^x_1 \) and

\[
S_k = (\psi, \sigma_0 e^{-H(n_2-n_1)} - \bar{P}(m_2-m_1) e \sigma_1 e \ldots e^{-H(n_k-n_{k-1})} - \bar{P}(m_k-m_{k-1}) e \sigma_1 \bar{\psi})
\]

Proof of Thm. 4.2. Let \( \tilde{H}(P) \)

\[
\tilde{H}(P) = \int_{-\pi}^{\pi} \epsilon(k) \xi(k) \bar{\xi}(k) \, dk
\]

Then eq. (4.1) can be written as

\[
S_k = (\psi, \sigma^x_1 e^{-H(n_2-n_1)} - \bar{P}(m_2-m_1) e \sigma^x_1 e \ldots e^{-H(n_k-n_{k-1})} - \bar{P}(m_k-m_{k-1}) e \sigma^x_1 \bar{\psi})
\]

Substituting \( - \) operators by \( \sim \) operators according to Thm. 3.2 i ii gives
The theorem follows since $U^{-1} \sigma_1^x = \sigma_1^x$ using Thm. 3.5 and $U = U^* = U^{-1}$.

Remark: If we are considering vacuum expectation values of Heisenberg operators (replace $n_j$ by $-i\mathbf{f}_j$) then the energy-momentum operators can be incorporated in $\sigma_1^x U$ by using the relation

$$e^{-iF} \hat{\xi}(k) e^{iF} = e^{i\mathbf{f}(k)} \hat{\xi}(k)$$

and its adjoint where

$$F = \int_{-\pi}^{\pi} f(q) \hat{\xi}^*(q) \hat{\xi}(q) dq,$$

$f$ real.

An infinite series representation for $S_k$ is obtained by inserting the spectral representation of $H$ and $P$ in eq. (4.1). We have

Thm. 4.3, For $T > T_c$

$$\hat{\xi}^*(f_{\mathbf{i}}) \hat{\psi} + \ldots + \hat{\xi}^*(f_{\mathbf{i}}) \hat{\psi} + \ldots n \text{ odd}$$

and

$$\hat{\psi}, \ldots, \hat{\xi}^*(f_{\mathbf{i}}) \hat{\psi} + \ldots \hat{\xi}^*(f_{\mathbf{i}}) \hat{\psi} + \ldots n \text{ even}$$

generate $\hat{A}_0$ and $\hat{A}_e$, respectively, where $\{f_{\mathbf{i}}\}$ is a complete orthonormal set in $L$. Thus

$$S_k = \sum \int dq_1^{\alpha_1} \int dq_1^{\beta_1} \ldots \int dq_1^{\alpha_{k-1}} (\hat{\psi} \sigma_1^x e^{-\gamma_{\alpha_1} \chi_{\alpha_1}^*}$$

$$(\chi_{\alpha_1} e^{-\gamma_{\beta_1} \chi_{\beta_1}^*}) \ldots (\chi_{\alpha_{k-2}} e^{-\gamma_{\alpha_{k-1}} \chi_{\alpha_{k-1}}} \chi_{\alpha_{k-1}} \sigma_1^x \hat{\psi})$$
where

$$\gamma_{\alpha_j} (\gamma_{\beta_j}) = (\sum_{i=1}^{n} c(q_i))^2 (n_{j+1} - n_j) + \hat{c} (\sum_{i=1}^{n} q_i) (m_{j+1} - m_j) \text{ for } \alpha_j = n \text{ (} \beta_j = n \text{)}$$

and

$$\chi_{\alpha_j} = (\alpha_j^{-1})^{1/2} \tilde{\xi}(q_1) ... \tilde{\xi}(q_{\alpha_j}) \tilde{\psi}, \alpha_j \text{ odd},$$

$$\chi_{\beta_j} = (\beta_j^{-1})^{1/2} \tilde{\xi}(q_1) ... \tilde{\xi}(q_{\beta_j}) \tilde{\psi}, \beta_j \text{ even}.$$ 

and the integral \( \int dq^\alpha (\int dq^\beta) \) extends over the \( \alpha (\beta) \) q variables occurring in \( \chi_\alpha \) and \( \chi_\beta \).

Similar results hold for \( T < T_c \) where

$$\tilde{\psi}, ..., \tilde{\xi}(f_{l_1}) \tilde{\xi}(f_{l_n}) \tilde{\psi}, ..., n \text{ even}$$

and

$$\tilde{\psi}, ..., \tilde{\xi}(f_{l_1}) \tilde{\xi}(f_{l_n}) \tilde{\psi}, ..., n \text{ even}$$

generate \( \hat{H}_\Omega \) and \( \hat{H}_\sigma \), respectively.

Remark: Using the explicit form of \( \tilde{\psi} \) in terms of \( \tilde{\xi} \), \( \tilde{\xi}^* \) and \( \tilde{\psi} \) the above matrix can in principle be evaluated by applying Wick's theorem. However in ref. 18 a generalization of Wick's theorem is proved and used to evaluate \( \chi_\alpha, \chi_\beta \).

Proof of Thm. 4.1: The proof is given for the two-point function. A similar argument holds for the \( k \geq 2 \) point functions. We begin by showing that both systems of point functions can be expressed in terms of an infinite set

$$\{ G(k_1, ..., k_n) \}_n^\infty, \{ F(k_1, ..., k_n) \}_n^\infty ;$$

the first set for the Schwinger functions, the second for the correlation functions. The functions within each set will be shown to satisfy a set of coupled integral equations identical in form to those described in 7. This will imply that the two expectations, call them \( <\bullet> \) and \( <\bullet> \) are related by \( <\bullet> = c <\bullet> \); \( c = 1 \) follows from \( <\sigma_1 \sigma_2> \) = \( <\sigma_1 \sigma_2> = 1 \).
We establish a useful representation of the two-point function of the periodic state as in \(^7\). From Cor. 2.5.1, assuming \(n_2 > n_1\), and inserting a complete set of energy-momentum eigenvectors in \(S_{M^2}\) we have

(a) \(T < T_c\)

\[
S_{M^2} = \sum_{n \text{ even}} \frac{1}{n^2} \left| \langle \tau_{n_1}^* \psi_S^+, \xi_{k_1}^* \ldots \xi_{k_n}^* \psi_S^- \rangle \right|^2.
\]

\[
\exp\left\{ -\left( \sum_{j=1}^n \epsilon(k_j) + \delta_M \right)n_2 - n_1 \right\} - i\left( \sum_{j=1}^n k_j \right)(m_2 - m_1)
\]

(b) \(T > T_c\)

\[
S_{M^2} = \text{(same as above with } n \text{ odd}).
\]

In these formulas,

\[
\delta_M = \frac{1}{2} \left[ \sum_{k \in E^+} \epsilon_k - \sum_{k \in E^-} \epsilon_k \right] \to 0 \text{ as } M \to \infty.
\]

The last assertion follows from

\[
\lim_{M \to \infty} \left( \sum_{k \in E^+} \epsilon_k(k) - M \int_{-\pi}^{\pi} \epsilon(q) \, dq \right) = 0.
\]

Let, as in \(^7\),

\[
F^M_{\mathcal{P}}((k)_1, n) = M^{n/2} e^{-in\pi/4} \exp\left(i \sum_{j=1}^n (k_j - \phi_{k_j}) \right) \langle \psi_{S^+}, \xi_{k_1} \ldots \xi_{k_n} \tau_{n_1}^* \psi_S^+ \rangle.
\]

and

\[
F_\mathcal{P}(k)_1, n = M^{n/2} e^{-in\pi/4} \exp\left(i \sum_{j=1}^n (k_j - \phi_{k_j}) \right) \langle \psi_{S^+}, \xi_{k_1} \ldots \xi_{k_n} \psi_S^+ \rangle.
\]

Following Abraham \(^7\) we use the notation \(\Delta_{k_1 \ldots k_n}(k)_{1, n}\) to mean the set

\[
\{k_1, \ldots, k_n\} / \{k_1, \ldots, k_n\}.
\]

Theorem 4.4. Ref.\(7\) If \(\theta(e^{ik}) = e^{2i\phi_k}\) then
\[ E_{M}^{n}(k_{1}, n) = \frac{1}{M} \sum_{k \in \mathbb{S}} \Theta(\tilde{e}k_{0}) F_{M}(k_{1}, n) + \]
\[ \sum_{j=1}^{n} (-1)^{j-1} \Theta(e^{ikj})^{-1} F_{M}(\Delta_{j}(k_{1}), n) \]

Prf: Follows immediately from the representation

\[ \tilde{t}^{x}_{1} = M^{-1/2} \sum_{k \in \mathbb{S}} \left[ e^{-i\pi/4} e^{i(k+\phi_{k})} \tilde{e}_{k} + e^{i\pi/4} e^{-i(k+\phi_{k})} \tilde{e}_{k}^{*} \right]. \]

Theorem 4.5. Let

\[ \tilde{n}(k, 1) = 2(1-e^{-i(k+\ell)})^{-1} (\Theta(e^{i\ell})^{-1}(\Theta(e^{i\ell})^{-1} - 1). \]

Then

\[ 2M^{-1} \sum_{k \in \mathbb{S}} F_{M}(k_{0}, n)(e^{-i(k_{0}-\ell)}^{-1} (1 + (\Theta(e^{i\ell})^{-1})^{-1}) = \]
\[ = \sum_{j=1}^{n} (-1)^{j-1} \tilde{n}(k, j) F_{M}(\Delta_{j}(k_{1}), n). \]

Prf: From

\[ (\psi_{-}^{k_{0}}, \xi_{k}, \ldots, \xi_{k_{1}}^{*} \psi_{+}^{k}) = 0 \]

together with the relation \( T < T_{c} \)

\[ \xi_{k} = S_{1} \xi_{k_{0}} + S_{2} \xi_{k_{0}}^{*} \]

we have

\[ 0 = \sum_{k \in \mathbb{S}} 2M^{-1} e^{-i(k_{0}-\ell)}^{-1} \cos(\phi_{k_{0}}^{-1})(\psi_{-}^{k_{0}} \xi_{k_{0}} + \ldots + \xi_{k_{1}}^{*} \psi_{+}^{k}) + \]
\[ + \sum_{k \in \mathbb{S}} 2M^{-1} e^{-i(k_{0}+\ell)}^{-1} \sin(\phi_{k_{0}}^{-1})(\psi_{-}^{k_{0}} \xi_{k_{0}} + \ldots + \xi_{k_{1}}^{*} \psi_{+}). \]

After eliminating \( \xi_{k_{0}}^{*} \) in the second term using anti-commutation the result follows by direct substitution. For \( T > T_{c} \) the relation between \( \xi_{k} \)
and $\xi_k$ does not contain the last term involving $\xi_0^*$ but still the above formula is obtained.

It has been argued in 7 that the functions $F_M$ and $F_M^*$ have limiting values $F$ and $F^*$, respectively, as $M \to \infty$, satisfying

$$F^*((k)_{1,n}) = (2\pi)^{-1} \int dk_0 \theta(e^{ik_0}) F((k)_{0,n}) +$$

$$+ \sum_{j=1}^{n} (-1)^{j-1} \theta(e^{ik_j})^{-1} F(\Delta_j(k)_{1,n}), \quad n \geq 1 \quad (4.2)$$

and

$$p \frac{1}{\pi} \int dk_0 \theta^{-1} (e^{-i(k_0 - k)^2})^{-1} (1 + \theta(e^{ik_0}) \theta(e^{ik_j})^{-1}) =$$

$$= \sum_{j=1}^{n} (-1)^{j-1} h(k_j, k) F(\Delta_j(k)_{1,n}), \quad n \geq 1 \quad (4.3)$$

In addition, the two-point function can be written as

(a) $T < T_c$

$$<\sigma(n_1, m_1) \sigma(n_2, m_2)> = \lim_{M \to \infty} S_{M2} =$$

$$= \sum_{n \text{ even}} (2\pi n!)^{-n} \int \ldots \int |F^*((k)_{1,n})|^2 \exp \{-n_2 - n_1\} \cdot$$

$$(\sum_{j=1}^{n} \epsilon(k_j)) - i(m_2 - m_1)(\sum_{j=1}^{n} k_j) \, dk_1 \ldots dk_n \quad (4.4)$$

(b) $T > T_c$, the sum on the RHS above is now over odd $n_1$'s.

Since we have not been able to confirm the argument in ref. 7, a proof of the above facts is presented in Appendix A.

Remarks: 1. Assume $T < T_c$. The $n=0$ term in (4.4) is $|F^*(\phi)|^2$, where $F^*(\phi) = i \langle \psi_{S-}, \tau_1^x \psi_{S+}^* \rangle$, $\psi_{S+}^* \equiv m^*$. Eq. (4.2) is replaced by

$$(2\pi)^{-1} \int dk_0 \theta(e^{ik_0}) F(k_0) = m^* \quad (4.5)$$
The functions $F^x((k)^{1,2n})$ for $m > 1$ are determined through (4.2) once we know the set $\{F((k)^{1,n})\}$ for odd $n$. These obey the system

$$YP\cdot,(k)^{1,n} = \sum_{j=1}^{n} (-1)^{j-1} \hbar(k_j,\cdot) F(\Delta_j(k)^{1,n})$$

where $Y: L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ is the bounded operator given by

$$(Yf)(q) = P \frac{1}{\pi} \int dk J(k) (e^{ikq} - 1)^{-1} \left(1 + \theta(e^{ik})\theta(e^{i2q})^{-1}\right).$$

It has been shown in (7) that $\dim \ker Y = 1$ if $T < T_c$ and $= 0$ if $T > T_c$. In addition, it has been proved that the system (4.3) has a unique antisymmetric solution once $F(k)$ is given. This function satisfies $YF = 0$ and therefore is therefore uniquely specified except for a constant, which is then fixed by (4.5).

2. $T > T_c$. In this case, $<\sigma(n_1,m_1)\sigma(n_2,m_2)>$ is determined from $\{F^x((k)^{1,n})\}$, $n$ odd, or $\{F((k)^{1,n})\}$ with even $n$. Since now $\dim \ker Y = 0$, these functions are uniquely specified once $F(\psi)$ is given. It is easy to show that the appropriate value is

$$F(\psi) = \lim_{M \to \infty} (\psi^-_S,\psi^+_S).$$

To show that

$$S_2 = <\sigma(n_1,m_1)\sigma(n_2,m_2)>$$

we proceed as follows:

(a) $T < T_c$. Inserting a complete set of energy-momentum eigenvectors

$$S_2 = |(\tilde{\psi}, \sigma_1^x \tilde{\psi})|^2 + \frac{1}{2!} \int \int \int \int dk_1dk_2 e^{-ik_1^2 + ik_2^2} e^{-\epsilon(k_1^2 + \epsilon(k_2^2)}(n_2-n_1) - \epsilon(k_1,k_2)(m_2-m_1)$$

. $|(\tilde{\psi}, \tilde{\xi}(k_1) \tilde{\xi}(k_2) \sigma_1^x \tilde{\psi})|^2 + \ldots$

(b) $T > T_c$

$$S_2 = \int dk e^{-\epsilon(k^2)}(n_2-n_1) - \epsilon k(m_2-m_1)$$

$$|(-\tilde{\psi}, \tilde{\xi}(k) \sigma_1^x \tilde{\psi})|^2 + \ldots$$

Let
Theorem 4.6.

\[
G^x((k), n) = (2\pi)^{n/2} e^{-in\pi/4} e^{-i(\sum_j \phi_j)} \langle \tilde{\psi}, \tilde{\xi}(k_n) \ldots \tilde{\xi}(k_1) \sigma_1 \psi \rangle
\]

and

\[
G((k), n) = (2\pi)^{n/2} e^{-in\pi/4} e^{-i(\sum_j \phi_j)} \langle \tilde{\psi}, \tilde{\xi}(k_n) \ldots \tilde{\xi}(k_1) \sigma_1 \psi \rangle.
\]

Prf: Follows from direct substitution of \( g = \tilde{\xi}^* (\tilde{\psi}) + \tilde{\xi} (\tilde{\sigma}) \), where \( g(k) = e^{i\pi/4} e^{-i\phi \sqrt{2\pi}} \) in the expression for \( G^x((k), n) \).

Theorem 4.7.

\[
\pi^{-1} \int dk_0 [e^{i(k_0 - \xi)} - 1]^{-1} G((k), n) \left( 1 + (e^{i\theta}) \theta (e^{i\theta})^{-1} \right) = \sum_{j=1}^n (\xi - j \theta) G((\xi - j \theta), n)
\]

Prf: Let \( g_1, \ldots, g_n \) be arbitrary \( L^1 \) functions and \( f \in C \) such that \( e^{i\phi} \) \( f \) is smooth. Then

\[
0 = (\tilde{\psi}, \tilde{\xi}(g_1) \ldots \tilde{\xi}(g_n) \tilde{\xi}(f) \psi) = \\
\int g_n(k_n) \ldots g_1(k_1) (T_1 f)(k_0) (\tilde{\psi}, \tilde{\xi}(k_n) \ldots \tilde{\xi}(k_1) \tilde{\psi}) dk_n \ldots dk_0 \\
+ \sum_{j=1}^n (\xi - j \theta) \int g_j(k_j) (T_2 f)(k_j) dk_j \\
\text{subject to} \frac{\prod_{k=1}^{n} g_k(k_k)}{\prod_{k \neq j} \tilde{\xi}(k_{\chi}) \psi} \tilde{\xi}(k_{\chi}) \psi \text{ for } \chi \neq j \]
Using the definition of $G$, together with
\[ e^{i\Phi_\kappa(T_1 f)(k)} = (2\pi)^{-1} P \int (1-e^{i(u-k)}(1+\theta(e^{iU})\theta(e^{iK})^{-1})(e^{iU}f(u)))du \]
and
\[ e^{-i\Phi_\kappa(T_2 f)(k)} = \\
= (2\pi)^{-1} P \int (1-\theta(e^{iU})\theta(e^{iK})^{-1})(1-e^{i(k+u)})^{-1}(e^{iU}f(u))du \]
one gets
\[ \int du (e^{iU}f(u)) P \int \frac{1}{2\pi} dk_0 G((k)_0,n)(1-e^{i(k_0-u)})(1+\theta(e^{iK_0})\theta(e^{iU}))^{-1} = \\
\sum_{j=1}^{n} (-1)^{j-1} \int du (e^{iU}f(u)) (1-\theta(e^{iU}))^{-1}(1-e^{i(k+j)})^{-1} \\
\times G(\Delta_j(k)_1,n) . \]
The theorem follows.

Assume $T < T_g$. The function $G(k)$ satisfies $YG = 0$ and $(2\pi)^{-1},$
\[ \int \theta(e^{iK})G(k)dk = (\tilde{\psi},\sigma^x \tilde{\psi}) . \]
This implies
\[ G^x((k), = (\tilde{\psi},\sigma^x \tilde{\psi}) / m^*, (n \text{ even}) \]
and therefore
\[ S_2 = |(\tilde{\psi},\sigma^x \tilde{\psi}) / m^*|^2 <\sigma(n_1,m_1)\sigma(n_2,m_2) > . \]
By evaluating the correlation functions at coincident points we conclude
\[ |(\tilde{\psi},\sigma^x \tilde{\psi}) / m^*|^2 = 1 , \]
hence
\[ S_2 = <\sigma(n_1,m_1)\sigma(n_2,m_2) > . \]
A similar argument holds for $T > T_g$. 

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5. DECOMPOSITION OF THE PERIODIC STATE

In addition to properties iii) and iv) of Thm. 3.2 enjoyed by \( U \) we have

**Thm. 5.1.** \( [\sigma\chi^\tau, U] = 0 \) and for \( T < T_c \)

\[
[e^{i m P} U, \sigma\chi^\tau] = [e^{i m H} U, \sigma\chi^\tau] = 0.
\]

**Prf. of Thm. 5.1.** As \( \sigma\chi^\tau = \sigma_1 + \sigma_2 \) it follows from Thm. 3.5 that

\[
U \sigma_1 U^{-1} = \sigma_1 \quad \text{or} \quad [\sigma_1, U] = 0.
\]

If (for \( T < T_c \))

\[
\tilde{\chi} = \frac{2 N}{\Omega} \sum_{\ell = 1} \tilde{\xi}^*(\ell, \ell) \tilde{\chi} \in \tilde{H}_e
\]

then

\[
e^{i m P} \tilde{\chi} = \exp(i m \int q \xi^*(q) \tilde{\xi}(q) dq) \tilde{\chi}.
\]

Furthermore, if

\[
\tilde{\chi} = \frac{2 N}{\Omega} \sum_{\ell = 1} \tilde{\xi}^*(\ell, \ell) \tilde{\psi} \in \tilde{H}_e
\]

then

\[
U^{-1} e^{i m P} U \tilde{\chi} = U^{-1} e^{i m P} \tilde{\chi} = U^{-1} \exp(i m \int q \xi^*(q) \tilde{\xi}(q) dq) \tilde{\chi}
\]

\[
= \exp(i m \int q \xi^*(q) \tilde{\xi}(q) dq) \tilde{\chi} = e^{i m P} \tilde{\chi}.
\]

Thus we have shown that

\[
U^{-1} e^{i m P} U \uparrow \tilde{H}_e = e^{i m P} \uparrow \tilde{H}_e
\]

and similarly one shows the equality for \( \tilde{H}_0 \) so that \([e^{i m P}, \sigma\chi^\tau] = 0\). Substituting \( H \) for \( P \) in the above arguments shows that \([e^{i m H}, \sigma\chi^\tau] = 0\).

Since \( U^2 = \mathbb{I} \) and thus \( U^* U = \frac{1}{2} \mathbb{L} \uparrow \frac{1}{2} \mathbb{L} \) are orthogonal projections and \( \mathbb{L}^2 = 0 \). Thus for \( T < T_c \) the subspace \( H_e = \mathbb{P}_e H \) reduce \( \sigma\chi^\tau \), \( e^{i m P} \) and \( e^{i m H} \). We have

**Thm. 5.2.** Let \( T < T_c \) and \( \psi^e = \sqrt{\mathbb{P}_e} \psi = \frac{1}{\sqrt{2}} (\psi_+ \pm \psi_-) \) so that \(|\psi^e| = 1\).
\((\hat{\psi}, \sigma_1 \hat{\psi})\) admits the decomposition

\[
(\hat{\psi}, \sigma_1 \hat{\psi}) = \frac{1}{2} (\psi_+, \sigma_1 \psi_+) + \frac{1}{2} (\psi_-, \sigma_1 \psi_-)
\]

where \(\sigma_1\) is a generic product

\[
\sigma_1 = \prod_j \sigma_j e^{-H|n_j| - i\tilde{P} n_j}
\]

and \((\psi_+, \sigma_1 \psi_+), (\psi_-, \sigma_1 \psi_-)\) are translationally invariant states. Furthermore, the decomposition is non-trivial since \((\psi_+, \sigma_1 \psi_+) = (\psi_-, \sigma_1 \psi_-) \neq 0.\) If \(\sigma_1\) has an even number of factors

\[
(\hat{\psi}, \sigma_1 \hat{\psi}) = (\psi_+, \sigma_1 \psi_+) = (\psi_-, \sigma_1 \psi_-) = (\hat{\psi}, \sigma_1 \hat{\psi})
\]

and for an odd number

\[
(\hat{\psi}, \sigma_1 \hat{\psi}) = 0, (\psi_+, \sigma_1 \psi_+) = (\hat{\psi}, \sigma_1 \hat{\psi}) = (\psi_-, \sigma_1 \psi_-)
\]

\((\psi_+, \sigma_1 \psi_+), (\psi_-, \sigma_1 \psi_-)\) have the clustering property; thus they are indecomposable.

Remarks:

1. By redefining \(\tilde{\psi}\), if necessary, \((\psi_+, \sigma_1 \psi_+) = (\hat{\psi}, \sigma_1 \hat{\psi})\) can be taken to be positive and is usually called the spontaneous magnetization.

2. Let \(H'\) denote the closure of the subspace of \(H\) generated by applying polynomials in \(\prod D_\ell \) (finite product), where \(\ell \in \mathbb{Z}\) is the set of integers and

\[
D_\ell = e^{-H|n_\ell|} e^{i n_\ell P} e^{-i n_\ell P, \sigma_1 e^{i n_\ell P}},
\]

to \(\tilde{\psi}\). Let \(H''\) be the physical Hilbert space obtained from the GNS construction applied to the state \((\tilde{\psi}, \hat{\psi})\). We expect, but have been unable to show that \(H' = H\) or \(H'' = H\).

Prf.: We have, using \(P_+ P_- = [P_+, \sigma_1] = 0\),

\[
(\hat{\psi}, \sigma_1 \hat{\psi}) = ((P_+ + P_-) \tilde{\psi}, \sigma_1 (P_+ + P_-) \tilde{\psi})
\]

\[
= (P_+ \tilde{\psi}, \sigma_1 P_+ \tilde{\psi}) + (P_- \tilde{\psi}, \sigma_1 P_- \tilde{\psi})
\]

\[
= \frac{1}{2} (\psi_+, \sigma_1 \psi_+) + \frac{1}{2} (\psi_-, \sigma_1 \psi_-)
\]
which shows the decomposition. If $\mathcal{O}$ has an even number of factors then

$\hat{H}_e \rightarrow \hat{H}_e$ and $\hat{H}_o \rightarrow \hat{H}_o$ so that

$$(\psi_\pm, 0 \psi_\pm) = \frac{1}{2} \left[ (\bar{\psi} \pm \bar{\psi}), (\bar{\psi} \pm \bar{\psi}) \right] = \frac{1}{2} \left[ (\bar{\psi}, 0 \bar{\psi}) + (\bar{\psi}, 0 \bar{\psi}) \right] = (\bar{\psi}, 0 \bar{\psi}) = (\bar{\psi}, 0 \bar{\psi}).$$

If $\mathcal{O}$ has an odd number of factors then

$$(\psi_\pm, 0 \psi_\pm) = \pm \frac{1}{2} \left[ (\bar{\psi}, 0 \bar{\psi}) + (\bar{\psi}, 0 \bar{\psi}) \right] = \pm (\bar{\psi}, 0 \bar{\psi}),$$

since

$$(\bar{\psi}, 0 \bar{\psi}) = (U\bar{\psi}, 0 V^{-1} \bar{\psi}) = (\bar{\psi}, U^{-1} U^{-1} \bar{\psi}) = (\bar{\psi}, 0 U^{-1} \bar{\psi}) = (\bar{\psi}, 0 \bar{\psi}).$$

If $\mathcal{O} = \sigma_1^x$ then $(\psi_\pm, \sigma_1^x \psi_\pm) = \pm (\psi, \sigma_1^x \bar{\psi})$ is shown to be non-zero in Appendix C which implies that the states are distinct. The clustering of these states is a consequence of the uniqueness of the vacuum in $H_\pm = \mathcal{P}_\pm H$ (which follows from the known double degeneracy of the ground state of $(H, \mathcal{P})$).

**APPENDIX A: THERMODYNAMIC LIMIT**

We remark that it can be shown that, for high temperatures, the sequence $\{S_{MK}\}$ converges to $S_k$, given by (4.1). For all temperatures compactness can be used to extract a convergent subsequence. In this appendix we show that any subsequence of $\{S_{Mk}\}$ where $M$ is of the for $2^k$ contains a convergent subsequence with limit given by (4.4) if $T < T_c$, or the appropriately modified (4.4) for $T > T_c$ where $F^2$ is given by (4.2) and $F$ satisfies (4.3). The idea of the proof is to show that the sequence $\{F_{M}^{2}\}$ is a normal family of analytic functions in the variable $z = e^{i \frac{\pi}{k}}$ in a neighborhood (independent of $M$) of $|z| = 1$. Then, we show that the limit $F$ of any convergent subsequence satisfies (4.3) and in addition the same subsequence of correlation functions converges to (4.4). We will assume $T < T_c$ but similar arguments apply for $T > T_c$ and $k > 2$.

Recall the definition
\[ F_M(k) = M^{1/2} e^{-i\pi/4} \exp(i(k-\xi)) \langle \psi^-_S, \xi \psi^+_S \rangle . \]

**Thm. A.1.** $F_M$ as a function of $z = e^{i\theta}$ extends as an analytic function in an annulus, independent of $M$ but dependent on $T$, around $|z| = 1$ and is uniformly (in $M$) bounded on compact subsets.

**Proof:** Using the relationship between $\xi$ and $\xi$, we write

\[ F_M(z) = e^{-i\pi/4} M^{-1/2} \sum_{\ell \in \mathbb{S}^+} e^{i(\ell+\phi)}/(z-e^{i\ell})^{-1} (e^{i\ell})^{-1} - \theta(z)^{-1} . \]

\[
\langle \psi^-, \xi_{-\ell} \psi^+_S \rangle .
\]

Let the annulus of analyticity be described by $r < |z| < R$, with $r < 1$ and $R > 1$. The function $(\vartheta^{-1}(\omega) - \vartheta^{-1}(z))/(z-\omega)$ is analytic in both variables in $r < |z|$, $|\omega| < R$ and therefore uniformly bounded in $r < r' < |z|$, $|\omega| < R'$, with $n < 1$ and $R' > 1$ independent of $M$. Let $C$ be the bound. Then for $r' < |z| < R'$,

\[ |F_M(z)| \leq CR^{-1/2} C \sum_{\ell \in \mathbb{S}^+} |\langle \psi^-_S, \xi_{-\ell} \psi^+_S \rangle| \leq CR' \left( \sum_{\ell \in \mathbb{S}^+} |\langle \psi^-_S, \xi_{-\ell} \psi^+_S \rangle|^2 \right)^{1/2} . \]

Since the set

\[
\{ \xi_{\ell_1} \psi^+_S \}_{\ell \in \mathbb{S}^+}, \{ \xi_{\ell_2} \xi_{\ell_3} \psi^+_S \}_{\ell_1}, \ldots ,
\]

\[ \ell_1 < \ell_2 < \ell_3 < \ldots \text{ form an orthogonal basis in } H, \]

we have

\[ l = \langle \psi^-_S, \psi^-_S \rangle = \sum_{\ell \in \mathbb{S}^+} |\langle \psi^-_S, \xi_{-\ell} \psi^+_S \rangle|^2 + \frac{1}{2} \sum_{\ell_1, \ell_2} |\langle \psi^-_S, \xi_{\ell_1} \xi_{\ell_2} \psi^+_S \rangle + \ldots |
\]

It follows that $|F_M(z)| \leq CR'$. A similar result holds for $F_M(z, \ldots, z_{2n+1})$. Thus, any subsequence of $M$ contains a subsequence for which $F_M(z, \ldots, z_{2n+1})$ converges uniformly on compact subsets of $r < |z| < R$. For notational convenience, we still call the converging subsequence $F_M$ and call the limit $F(z, \ldots, z_{2n+1})$. We next show that $F$ satisfies (4.3). Since $M$ has
the form $2^n$ this implies $S^+_M \subset S^+_M$ if $M' > M$. Let $R^- = \bigcup_M S^+_M$, and let $h(z)$ be an arbitrary analytic function on $|z| < R$. If

$$g_M(k) = 2M^{-1} \sum_{\lambda \in S^+_M} h(e^{i\lambda})(e^{i(k-\lambda)} - 1)^{-1} (1 + \theta(e^{i\lambda})/\theta(e^{i\lambda})), k \in R^-,$$

then

$$\lim_{M \to \infty} g_M(k) = g(k) = \pi^{-1} \int_{-\pi}^{\pi} \frac{h(e^{i\lambda})(e^{i(k-\lambda)} - 1)^{-1} (1 + \theta(e^{i\lambda})/\theta(e^{i\lambda})) d\lambda}{2\pi},$$

uniformly in $k \in R^-$. 

Proof: Write

$$g_M(k) = 2M^{-1} \sum_{\lambda \in S^+_M} h_1(e^{i\lambda})(e^{i(k-\lambda)} - 1)^{-1} + \theta(e^{i\lambda})2M^{-1} \sum_{\lambda \in S^+_M} h_2(e^{i\lambda})(e^{i(k-\lambda)} - 1)^{-1}, \quad (A.1)$$

where

$$h_1(e^{i\lambda}) = h(e^{i\lambda}) \quad \text{and} \quad h_2(e^{i\lambda}) = \theta(e^{i\lambda})^{-1} h(e^{i\lambda}).$$

Let

$$B^+_k = \{ \lambda \in S^+: \text{Im} e^{i(k-\lambda)} > 0 \}.$$ 

Then,

$$2M^{-1} \sum_{\lambda \in S^+_M} h_1(e^{i\lambda})/(e^{i(k-\lambda)} - 1) =$$

$$= 2M^{-1} \sum_{\lambda \in B^+_k} \left[ h_1(e^{i\lambda})(e^{i(k-\lambda)} - 1)^{-1} + h_1(e^{i(2k-\lambda)})(e^{i(k-\lambda)} - 1)^{-1} \right] =$$

$$= 2M^{-1} \sum_{\lambda \in B^+_k} \left[ h_1(e^{i\lambda}) - h_1(e^{i(k-\lambda)}) \right]/(e^{-i\lambda} - 1).$$

Consider the function of two variables

$$(ah_1(\omega) - h_1(\omega/2))/|1-z|, \quad z \neq 1;$$

$$-h_1(\omega) + 2\omega h_1'(\omega), \quad z = 1.$$
This is analytic in \( r < |\omega|, \ |z| < R, \) and in particular uniformly continuous for \( |\omega| = |z| = 1. \) From this, it follows that

\[
\lim_{M \to \infty} 2M^{-1} \sum_{k \in \mathcal{S}^+} h_1(e^{ik\omega})(e^{ik\omega} - 1)^{-1} = \\
= \pi^{-1} \int_{-\pi}^{\pi} \frac{[e^{ik\omega} h_1(e^{ik\omega}) - h_1(e^{ik\omega})] d\omega}{1 - e^{ik\omega}} \\
= \pi^{-1} \int_{-\pi}^{\pi} h_1(e^{ik\omega})(e^{ik\omega} - 1)^{-1} d\omega,
\]

uniformly in \( k \in \mathcal{S}^+. \) The second term in (A.1) is treated similarly and the proof is complete.

**Thm. A.3:**

\[
(YF)(k) = \pi^{-1} \int \frac{d\omega}{-\pi} F(k)(e^{ik\omega} - 1)^{-1} (1 + \theta(e^{ik\omega})/\theta(e^{ik\omega})) = 0. 
\]

**Proof:** Since

\[
2M^{-1} \sum_{k \in \mathcal{S}^+} F_M(k)(e^{ik\omega} - 1)^{-1} (1 + \theta(e^{ik\omega})/\theta(e^{ik\omega})) = 0 \quad (k \in \mathcal{S}^+),
\]

we have, upon multiplying by \( 2M^{-1} h_1(k) \) and summing over \( R \in \mathcal{S}^+ \),

\[
2M^{-1} \sum_{k \in \mathcal{S}^+} F_M(k) g_M(k) = 0.
\]

Using the uniformity of convergence of \( F_M \) and \( g_M \), it is easy to show that

\[
0 = \lim_{M \to \infty} 2M^{-1} \sum_{k \in \mathcal{S}^+} F_M(k) g_M(k) = \lim_{M \to \infty} 2M^{-1} \sum_{k \in \mathcal{S}^+} F(k) g(k) = \\
= \pi^{-1} \int F(k) g(k) d\omega.
\]

We see from Lemma A.2 that \( (\overline{F}, \overline{Gk}) = 0, \) where the operator \( \overline{F} \) is defined by \( \overline{Ff} = (\overline{f}) \). Using self-adjointness of \( \overline{F} \) and the fact that analytic functions on a neighborhood of \( |z| = 1 \) are dense in \( L^2(\text{torus}) \) we conclude that \( YF = 0. \)
Using the same argument, it is easy to show that
\[ p^{-1} \int_{-\pi}^{\pi} F((k)_{0,n}) (e^{i(k_{0}-k)} - 1)^{-1} (1 + \theta(e^{i(k_{0}-k)}) \theta(e^{i\lambda})) dk_{0} = \]
\[ = \sum_{j=1}^{n} (-1)^{j-1} \ h(k_{0}, \lambda) F(\Delta_{j}(k),_{1,n}) \]
f\est for all \( k, \ldots, k_{n} \in \mathbb{R}^{-} \) and then for all \( k_{i} \) by continuity. Thus, we see that the set of limiting functions \( F \) satisfies (4.3).

Recall from section 4 that \( (T < T_{0}, n_{2} > n_{1}) \)
\[ S_{M} = \sum_{n \text{ even}} \cdots = \sum_{n} \alpha_{n_{2}-n_{1}, m_{2}-m_{1}}^{M} \]
where
\[ \frac{F_{M}^{\pi}}{F_{M}^{\pi}}((k)_{1,n}) = M^{-1} \sum_{\kappa_{0} \in \mathbb{S}} \theta(e^{i\kappa_{0}}) F_{M}((k)_{0,n}) + \]
\[ + \sum_{j=1}^{n} (-1)^{j-1} \theta(e^{i\lambda}) F_{M}(\Delta_{j}(k),_{1,n}) \]
(see Thm. 4.4). From the results above we see that \( F_{M}^{\pi} \) converges uniformly in \( k_{i} \in \mathbb{R}^{-} \) to
\[ F^{\pi}((k)_{1,n}) = (2\pi)^{-1} \int_{-\pi}^{\pi} \theta(e^{i\kappa_{0}}) F((k)_{0,n}) dk_{0} + \]
\[ + \sum_{j=1}^{n} (-1)^{j-1} \theta(e^{i\lambda}) F(\Delta_{j}(k),_{1,n}) . \]
This last formula allows a continuous extension of \( F^{\pi} \) to all \( k_{i} \in [-\pi, \pi] \).
It follows that
\[ \lim_{M \to \infty} M^{-1} \sum_{k_{0} \in \mathbb{S}} | \frac{F_{M}^{\pi}}{F_{M}^{\pi}}((k)_{1,n}) |^{2} \]
\[ \times \exp[-(n_{2}-n_{1})( \sum_{j=1}^{n} \epsilon(k_{j})) - \frac{m_{2}-m_{1}}{n_{2}-n_{1}}( \sum_{j=1}^{n} \epsilon(k_{j})) - m_{2}-m_{1}] \]
\[ = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} | F^{\pi}((k)_{1,n}) |^{2} \exp[-(n_{2}-n_{1})( \sum_{j=1}^{n} \epsilon(k_{j})) \]
\[ - \frac{m_{2}-m_{1}}{n_{2}-n_{1}}( \sum_{j=1}^{n} k_{j})] \ dk_{1} ... dk_{n} \equiv n! \alpha_{n,n_{2}-n_{1}, m_{2}-m_{1}} . \]
Thm. A.4. There exists a constant $c$ (independent of $M$) such that

$$\alpha_{n,0,0}^M = c^{n+1}(n+1)!.$$ 

Corollary A.5. \(\lim_{M \to \infty} \frac{S_2}{M^2} = \sum_{n \text{ even}} a_{n,n_2-n_1,m_2-m_1}.

Proof: Follows from the dominated convergence theorem.

Proof of Thm. A.4: Starting from the definition of $F_M(k_1,n)$ we get

$$|F_M|_M^2 = M^{-n} \sum_{k_1 \in \mathcal{S}} |F_M(k_1,n)|^2 = \sum_{k_1 \in \mathcal{S}} \prod_{i=1}^n (\psi_{S+}, \xi_{k_1}^* \psi_{S+})^2 \leq \sum_{k_1 \in \mathcal{S}} \prod_{i=1}^n (\psi_{S+}, \xi_{k_1}^* \psi_{S+}).$$

The right hand side can be calculated by Wick's theorem. Terms involving contractions like $(\psi_{S+}, \xi_{k_1}^* \xi_{k_1} \psi_{S+})$ or $(\psi_{S+}, \xi_{k_1}^* \xi_{k_1}^* \psi_{S+})$ do not contribute because of the summation over the $k_1$s. Thus we see that

$$|F_M|_M^2 \leq \sum_{k_1 \in \mathcal{S}} \prod_{i=1}^n (-1)^\ell (\psi_{S+}, \xi_{k_1}^* \psi_{S+}) \ldots (\psi_{S+}, \xi_{k_1}^* \psi_{S+}) = \sum_{k_1 \in \mathcal{S}} \det A

where the self-adjoint matrix $A$ has elements

$$A_{i,j} = (\psi_{S+}, \xi_{k_1}^* \xi_{k_1}^* \psi_{S+}).$$

We calculate, using $\xi_k = \frac{1}{2} k \xi_k + \frac{1}{2} k^2 \xi_k^*$,

$$A_{i,j} = 4M^{-2} \sum_{\ell \in \mathcal{S}} e^{-i(\ell-k_i)} (1-e^{-i(\ell-k_i)}) e^{-i(\ell-k_j)} (1-e^{-i(\ell-k_j)}) e^{-i(\ell-k_i^*)} \sin(\phi_{k_1}^* - \phi_{k_1}) \sin(\phi_{k_1} - \phi_{k_1}).$$

Let

$$c_1 = \frac{|\sin(\phi_{k_1} - \phi_{k_1})/(1-e^{-i(\ell-k_i^*)})| < \infty}{\ell, k \in [-\pi, \pi]}$$

(c, independent of $M$). Then $|A_{i,j}| \leq 4c_1^2/M$ and by Hadamard's inequality.
\[ |\det A| \leq (4a^2/M)^n \cdot n^{n/2} \] and therefore \[ |E_M|_M^2 \leq (4a^2_1)^n \cdot n^{n/2} \equiv c_2^n \cdot n^{n/2}, \]
where \( c_2 = 4a^2_1 \). Now, majorizing \( F^\infty \), we have
\[ |E_M|_M \leq (c_2^{n+1}(n+1)/2)^{1/2} + n(c_2^{n-1}(n-1)/2)^{1/2}. \]
This implies
\[ |E_M|_M^2 \leq 2(c_2^{n+1}(n+1)/2) + n^2(c_2^{n+1}(n+1)/2), \quad c_3 = \max(1, c_2), \]
\[ |E_M|_M^2 \leq 4(\sigma + 1)^2 c_3^{n+1}(n+1) + 2 \leq \sigma^{n+1}(n+5), \quad \sigma = 4 \cdot c_3, \]
and completes the proof of the theorem.

**APPENDIX B**

The structure of \( U \) and \( \tilde{\psi} \) depends on the null space of \( T_1 \) which in turn depends on the temperature by Thm. 3.1 vii. Let \( \eta_0(\cdot, \cdot) \) denote the normal ordering operation with the \( \hat{\psi} \) vacuum projection occurring after the last creation operation. For \( T > T_c \) we have

Thm.B.1.

a) \( \tilde{\psi} = (\det T_1^{t_1})^{1/4} \exp(-\frac{1}{2} \int \hat{\xi}^*(k) (T_1^{-1} T_1) (k, k') \hat{\xi}^*(k') d\mathbf{k} d\mathbf{k'}) \)

\[ \tilde{\psi}_\psi \psi = (\det T_1^{t_1})^{1/4} \eta_0 (\exp \{ -\frac{1}{2} \int (\hat{\xi}(k), \hat{\xi}^*(k)) \} \]

b) \( U = (\det T_1^{t_1})^{1/4} \eta_0 (\exp \{ -\frac{1}{2} \int (\hat{\xi}(k), \hat{\xi}^*(k)) \} \)

\[ \begin{pmatrix} T_2 \quad T_1^{-1} \\ -T_1^{-1} \quad T_2 \end{pmatrix} \begin{pmatrix} T_1^{-1} \quad T_1 \\ T_1 \quad T_2^{-1} \end{pmatrix} \begin{pmatrix} \hat{\xi}(k) \\ \hat{\xi}^*(k') \end{pmatrix} = \begin{pmatrix} \hat{\xi}(k') \\ \hat{\xi}^*(k) \end{pmatrix} \]

(8.2)

\[ \frac{d\mathbf{k} d\mathbf{k'}}{d\mathbf{k} d\mathbf{k'}} \]

\[ \tilde{\psi}_\psi \psi = \hat{H}_\psi \hat{H}_\psi = \hat{H}_\psi \hat{H}_\psi = \hat{H}_\psi \]

Remark: 1. By expanding the exponential in (8.2) we see that

\[ (\hat{\psi}, \hat{\xi}(k_1) \hat{\xi}(k_2) \hat{\psi}) / (\hat{\psi}, \hat{\psi}) = (T_1^{t_1}) (k_1, k_2) = (T_2^{t_1}) (k_1, k_2) \]
and
\[ (\tilde{\xi}(q) \tilde{\psi}, U \tilde{\xi}(k) \tilde{\psi}) = T_1^{-1}(q, k) . \]

These functions are obtained explicitly in ref. 18 by Wiener–Hopf methods.

2. With our convention for \( \tilde{\xi}(f) \) we associate the matrix

\[ J = \begin{pmatrix} T_1^t & T_2^t \\ T_1 & T_2 \end{pmatrix} \]

with the transformation eqs. (2.1a,b) which differs from that of ref. 12. Our operators are the transpose of those in ref. 12. If \( B, D \) are matrices associated with pict then BD corresponds to the transformation first C followed by B, and in terms of the corresponding unitaries \( U_{BD} = U_B U_D \).

Proof: a), b) follow from\(^\text{12}\) since \( \dim \ker T_1 = 0 \).

\( c) \ (T_1^{-1} T_2)^* = T_2 T_1^{-1} \) and \( T_1^* = T_1 \) imply \( U = U^* \).

Take \( T < T_c \) in what follows.

Lemma 8.2. The matrix

\[ J = \begin{pmatrix} T_1^t & T_2^t \\ T_1 & T_2 \end{pmatrix} \]

associated with the pict of eqs. (2.1a,b) is the product of the three pict \( v_1, a \) and \( v_2 \), i.e.

\[ J = v_2 a v_1 = \begin{pmatrix} V_2^t & V_1' \psi' & V_2' \psi' & V_2' \phi' \\ V_2' \psi' & V_1 & V_2 & v_1' \phi' \end{pmatrix} \]

where

\[ v_1 = \begin{pmatrix} V_1^t & 0 \\ 0 & V_1 \end{pmatrix}, \quad a = \begin{pmatrix} \phi' & \psi' \\ \psi' & \phi' \end{pmatrix}, \quad v_2 = \begin{pmatrix} V_2^t & 0 \\ 0 & V_2 \end{pmatrix} \]

and \( V_1, V_2 \) are the unitary operators defined by

\[ (V, \phi)(k) = \sqrt{c} \ e^{-ik/2} \phi(k) \quad (B.4) \]

\[ (V_2 \phi)(k) = \sqrt{c} \ e^{-iq/2} \phi(-q) = \sqrt{c} \ e^{-iq/2} (V \phi)(q) \quad (B.5) \]
is the integral operator

\[
\hat{\phi} \hat{\phi}(k) = \pi^{-1} p \int \frac{1}{2\pi} (2-2 \cos(k+q))^{-1/2} \epsilon(k+q) \cos(\phi_\hat{\phi}(k) + \phi_\hat{\phi}) \phi(q) dq,
\]

\[
\hat{\phi} = \hat{\phi}^* = \hat{\phi}, \quad \text{dim ker } \hat{\phi} = 1
\]

and \( \psi = V_1^* T_2 V_2^* \), \( \epsilon(x) = \pm \) if \( x > 0 \).

Proof:

Using the group property of plct and the fact that \( T_1 \) can be written as

\[
(T_1 \hat{f})(k) = \int \sqrt{\epsilon} e^{-ik/2} \frac{1}{\pi} (2-2 \cos(k-q))^{-1/2} \epsilon(k-q) \cos(\phi_\hat{\phi}(k) + \phi_\hat{\phi}) (\sqrt{\epsilon} e^{i\hat{q}/2}) \hat{f}(q) dq
\]

\[
= \int T_1(k,q) \hat{f}(q) dq = \int T_1(k,-q) \hat{f}(q) dq
\]

the decompositions for \( T_1 \) and \( \mathcal{J} \) follow.

As \( \phi = \phi^* = \overline{\phi} \) and \( \text{dim ker } \phi \neq 0 \) the plct a of Lemma B.2. is a special case \( \text{dim ker } \phi = 1 \) of a decomposable plct in the sense of ref. 12:

1. \( L = L_1 \oplus L_2 \) where \( L = \text{ker} \text{O} \). \( L_2 \) are invariant under \( \phi, \psi, \overline{\psi}, \psi^* \) thus \( \phi = \phi_1 \oplus \phi_2, \psi = \psi_1 \oplus \psi_2, \psi_1 \) is unitary on \( L_1 \). \( \phi_2^{-1} \) is bounded on \( L_2 \).

2. The transformation

\[
\hat{b}(\phi_1) = a(\phi_1) + a^*(\phi_2)
\]

\[
\hat{b}^*(\phi_1) = a(\overline{\phi_2}) + a^*(\phi_2)
\]

\( i = 1, 2, \phi = \phi_1 + \phi_2 \), is a plct in \( \mathcal{H} \), the subspace of \( \mathcal{H} \) generated by \( \hat{a}^*(\phi), \phi \in L_i \), with matrix \( \begin{pmatrix} \phi_1 & \psi_1 \\ \overline{\psi_2} & \overline{\phi_2} \end{pmatrix} \).

Thm. B.3. Let \( \phi_1 = \alpha_1, h = \overline{h} \in \text{ker } \phi, |h| = 1 \). Let \( \mathcal{F}_0(F) \) be the vacuum vector for the a's \( (b_i)'s \) of eq. (B.7a,b) satisfying \( |F| = |F_0| = 1 \), and let \( \mathcal{W} \) be the unitary which implements a. Then
a) \( \psi_1 \phi_1 = \gamma \phi_1 \) with \( |\gamma| = 1 \),

b) \( F = R_1 R_2 F_0 \)

where \( R_1 = a^*(h) \) and \( R_2 F_0 \) is given by eq. (B.1) with \( \phi_2, \psi_2, F \). \( a^*(k) \) replacing

\[ T_1', T_2', \hat{\psi}, \xi^*(k). \]

c) Let \( a, \beta \) satisfy \( |a| = |\beta| = 1 \), \( \beta = a \gamma \), then

\[ W = W_1 W_2, \]

where \( W_1 = a(\alpha h) + a^*(\beta h) \) and \( W_2 \) is given by eq. (B.2) with \( -\phi_2, -\psi_2, a(q), a^*(q) \) replacing \( T'_1, T'_2, \hat{\xi}(q), \xi^*(q) \) and \( N_0 \) referring to the \( a \)'s.

Proof: b) First we show \( b(\phi) F = 0 \) for all \( \phi \),

\[ b(\phi) = b(\phi_1)+b(\phi_2), \ b(\phi_1) = a^*(\psi_1) = \gamma a^* \]

so that \( b(\phi_1) F = 0 \) since \( b(\phi_1) R_1 = 0 \). Now

\[ b(\phi_2) F = -R_1 \ b(\phi_2) R_2 F_0 \ as \ \{ b(\phi_2), R_1 \} = 0 \]

and \( b(\phi_2) R_2 F_0 \) is zero by Thm. B.1. c). \( W_1 \) satisfies \( b(\phi_1) W_1 = W_1 a(\phi_1) \),

\[ b^*(\phi_1) W_1 = W_1 a^*(\phi_1) \] and \( W_2 \) satisfies \( -b(\phi_2) W_2 = W_2 a(\phi_2) \), \( \ b^*(\phi_2) W_2 = = W_2 a^*(\phi_2) \) and as \( b(\phi_2) W_1 = -W_1 b^*(\phi_2) \) we have

\[ b(\phi) W = (b(\phi_1) + b(\phi_2)) W_1 W_2 \]

\[ = W_1 (a(\phi_1) - b(\phi_2)) W_2 \]

\[ = W_1 W_2 (a(\phi_1) + a(\phi_2)) = W_1 a(\phi) . \]

A similar argument shows that \( b^*(\phi) W = W_1 a^*(\phi) \).

Thm. B.4. Let \( U_1, U_2, W_a = W a_1 W a_2, U = U_2 W a U_1 \) be the unitaries corresponding to the pict \( v_1, v_2, a, J \). Then

a) \( \hat{\psi} = U \hat{\psi} = U_2 R_1 R_2 \hat{\psi} \)

and

\[ U = U_2 W a_1 W a_2 U_1. \]
where $U_1$, $U_2$ are the usual Fockified operators of $V$, and $V_2$: $R_1$, $R_2$, $W_{a_1}$, $W_{a_2}$ are as in Thm. 8.3. with $U_1 \tilde{\xi}(k)U_1^{-1}$, $U_1\tilde{\xi}^*(k)U_1^{-1}$ replacing $a(k)$, $a^*(k)$ and $W_{a_1}$ replacing $W_{a_2}$.

b) $\tilde{\psi} \in \hat{H}_o$, $\tilde{\nu}_e = \hat{H}_o$ and $\tilde{\nu}_e = \hat{H}_o$.

Proof: a) Follows from Lemma 8.2 and Thm. 8.3 since the unitaries corresponding to the pcpl give a ray representation of the pcpl group. b) Holds since $U_{2} : \hat{H}_e(o) \rightarrow \hat{H}_e(o)$ and $\hat{W} : \hat{H}_e(o) \rightarrow \hat{H}_o(e)$.

APPENDIX C: $(\tilde{\psi}, \sigma_{1}^{x} \tilde{\psi}) \neq 0 \text{ for } T < T_{o}$.

$$(\tilde{\psi}, \sigma_{1}^{x} \tilde{\psi}) = (\tilde{\psi}, (\tilde{\xi}^*(h) + \tilde{\xi}^*(\tilde{h})) \tilde{\psi}) = (\tilde{\psi}, \tilde{\xi}^*(h) \tilde{\psi})$$

where $\tilde{h}(k) = e^{i\pi/4} e^{-i\phi k/2\pi}$. $\tilde{\psi}$ is given by Thm. B.4 or $\tilde{\psi} = \omega/|\omega|$ where

$$\tilde{\omega} = \sum_{n \text{ odd}} \tilde{\xi}^{*}(n) \sigma_{n} \tilde{\psi},$$

and $c_{n}$ is an anti-symmetric $L^2([-\pi, \pi]^{n})$ function.

$$\tilde{\xi}(f) \tilde{\omega} = 0 \text{ for all } f \in L \text{ implies } (T_{o}, f, \tilde{c}_{1}) = 0,$$

or $c_{1} \in \ker T_{1}$, and we have

$$(\tilde{\psi}, \sigma_{1}^{x} \tilde{\psi}) = |\omega|^{-1} \int \tilde{\sigma}_{1}(x) \tilde{h}(x) dx = |\omega|^{-1} (\sigma_{1}, h).$$

Letting $g = e^{i\phi} \tilde{c}_{1}$ the requirement that $\tilde{c}_{1} \in \ker T_{1}$ becomes

$$T_{1}(e^{-i\phi} g) = \frac{1}{2} (\Theta g + \Theta Hg) = 0, \text{ } \Theta = e^{-2i\phi} g \equiv \Theta(e^{i\phi})$$

or passing to Fourier coefficients

$$\sum_{n=0}^{\infty} \Theta_{n-m} g_{m} = 0 \quad n \geq 0,$$

$$\sum_{n=0}^{\infty} \Theta_{n-1} g_{-(m+1)} = 0 \quad n \geq 0,$$
where
\[ g(e^{i\omega}) = \sum_{n=-\infty}^{\infty} g_n e^{i\omega}, \quad \Theta_{nm} = \Theta_{n-m} = \int e^{-i(n-m)\omega} g(e^{i\omega}) d\omega. \]

By using Wiener-Hopf factorization (see Krein\(^{14}\)) we find
\[ g(z) = \frac{(z - x_2^{-1})^{1/2}}{(z - x_1)^{1/2}} \]
and it is confirmed that
\[ \langle c_1, h \rangle = \int g(\lambda) e^{-2i\phi(x)} d\lambda \neq 0. \]

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REFERENCES


Resumo

Mostramos neste trabalho que as funções de Schwinger associadas as funções correlação do modelo de Ising bidimensional com condições de fronteira periódicas no limite termodinâmico podem ser representadas por uma fórmula de Feynmann - Kac (F-K) num espaço de Fock fermiônico H. Os operadores de campo e energia-momentum são expressos em termos de dois conjuntos de campos livres fermiônicos agindo em H. Estes dois conjuntos estão relacionados entre si por uma transformação linear canônica própria (pclt), i.e. existe um operador unitário U que implementa a transformação. Representações em séries para as funções de Schwinger são obtidas substituindo as representações espectrais dos operadores de energia-momentum na fórmula de F-K. Abaixo da temperatura crítica \( T_c = (I+\mu)/\beta \) são projeções ortogonais que comutam entre si e que reduzem a álgebra de observáveis, fornecendo a decomposição explícita do estado periódico em dois estados puros, invariantes por translação.