Exact Kantowski-Sachs and Bianchi Types I and III Cosmological Models with a Conformally Invariant Scalar Field

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Abstract Exact solutions of the Einstein-Conformally Invariant Scalar Field Equations are obtained for Kantowski-Sachs and Bianchi types I and III cosmologies. The presence of the conformally invariant scalar field is responsible for some interesting features of the solutions. In particular it is found that the Bianchi I model is consistent with the big-bang theory of cosmology.

1. INTRODUCTION

In the last three decades a respectable effort was devoted to the investigation of the field equations of the general theory of relativity for spatially homogeneous but anisotropic space-times. These space-times belong either to the Bianchi types I-IX or to the Kantowski Sachs class and are generally interpreted as cosmological models. Recently, the study of spatially homogeneous anisotropic cosmological models with a conformally invariant scalar field as the matter field has received some attention\(^1\)-\(^3\). The properties of the energy-momentum tensor of the conformally invariant scalar field are quite distinct from those concerning the ordinary scalar field\(^4\)-\(^6\). Besides, the predictions of the theories involving the two tensors are rather different in strong gravitational fields\(^7\).

In this paper exact solutions, given in a unique parametrisation, for Kantowski-Sachs (KS) and Bianchi types I and III cosmological models with a conformally invariant scalar field are obtained. An interesting feature of the former models is that one can recover from them the corresponding vacuum solutions in a straightforward way. The cosmological implications of the Bianchi I model are discussed in a systematic way.
2. FIELD EQUATIONS AND SOLUTIONS

In choosing local orthonormal bases $\sigma_{(a)}^{\mu}$, the KS metric ($a=1$), the Bianchi type \( I \) metric ($a=2$), and the Bianchi type \( I \) metric with two anisotropic directions ($a=3$) can be put in the form.

$$d^2s_{(a)}^2 = \eta_{\mu\nu} \sigma_{(a)}^{\mu} \sigma_{(a)}^{\nu}, \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1),$$  \hspace{1cm} (1)

where

$$\sigma_{(a)}^{0} = dt, \quad \sigma_{(a)}^{1} = x(t), \quad \sigma_{(a)}^{2} = y(t), \quad \sigma_{(a)}^{3} = \theta(t).$$

From the field equations:

$$R_{\mu\nu} f(S) = \delta_{\mu\nu} \theta S \theta S - 4 \theta S \theta S + 2 S \theta \theta S + 2 S \theta \theta S S ,$$

\hspace{1cm} (2a)

\hspace{1cm} (2b)

\hspace{1cm} (2c)

where $V$ denotes covariant differentiation and $f(S) = 1 - S^2, \quad S = (n/6)^{1/2} \phi$, \hspace{1cm} (2d)

$\phi$ being the massless scalar field, we obtain the following set of equations:

$$R_{00} = \frac{\ddot{x}}{x} + 2 \frac{\ddot{y}}{y} f = -3 \dot{S}^2 - 2S \dot{S} \frac{\ddot{x}}{x} + 2 \frac{\ddot{y}}{y} ,$$

\hspace{1cm} (3a)

$$R_{11} = \frac{\ddot{r}}{r} + 2 \frac{\ddot{z}}{z} f = \ddot{S}^2 + 2S \dot{S} \frac{\ddot{r}}{r} ,$$

\hspace{1cm} (3b)

$$R_{22} = R_{33} = \frac{\ddot{u}}{u} + \frac{\ddot{v}}{v} \left( \frac{\ddot{u}}{u} + \frac{\ddot{v}}{v} \right), \quad S = K_1 / xy^2 ,$$

\hspace{1cm} (3c)

where dots, as usual, denote derivative with respect to $t$, $\&=+1,-1$, or

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0, according to whether $a=1$, $a=2$ or $a=3$ is chosen, and $K_1$ is an integration constant.

Introducing the new time variable $T$ by $dt = y^2 dT$, the linear combination of (3b) and (3c) gives

$$\left(xy'\right)' + K_1 xy = 0,$$

(4a)

where $(\cdot)' = \frac{d}{dT}$. From (3b) one gets

$$\left(x'y - K_1 \mathcal{S}\right)' = 0$$

(4b)

In addition one has the constraint equation

$$\left(xy\right)'/xy + (y'/y)' - x'y'/xy + \varepsilon = 0.$$  

(4c)

The solutions of the set of eqs. (4) can be presented in the form

$$E = +1:\quad x = \left[\tan \frac{T}{2}\right]^{\alpha - \beta} + \left[\tan \frac{T}{2}\right]^{\beta + \alpha - \beta} = 1;$$

$$y = \left(\sin T\right) \left[\tan \frac{T}{2}\right]^{\alpha + \beta} + \left[\tan \frac{T}{2}\right]^{\alpha - \beta},$$

$$\frac{1+S}{1-S} = \left[\tan \frac{T}{2}\right]^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1;$$

(5a)

$$E = -1:\quad x = \left[\tanh \frac{T}{2}\right]^{\beta - \alpha} + \left[\tanh \frac{T}{2}\right]^{\alpha + \beta},$$

$$y = \left[\sinh T\right] \left[\tanh \frac{T}{2}\right]^{\alpha + \beta} + \left[\tanh \frac{T}{2}\right]^{\alpha - \beta},$$

$$\frac{1+S}{1-S} = \left[\tanh \frac{T}{2}\right]^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1;$$

(5b)

$$E = 0:\quad x = T^{\beta - \alpha} + T^{\beta + \alpha};$$

$$y = T^{1-\alpha - \beta} + T^{1+\alpha - \beta},$$

$$\frac{1+S}{1-S} = T^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1.$$  

(5c)
For convenience, the multiplicative constants, as well as an additive integration constant concerning the variable $T$ have been eliminated.

3. DISCUSSION

Solutions (5a) and (5b) are new. The first one represents a closed spatially homogeneous anisotropic cosmological model, whereas the second belongs to an open universe. A universe is said to be closed or open, according to whether the space-time presents spatially homogeneous sections with compact or noncompact manifolds structures, respectively. Traditionally, the open or closed feature of a cosmological model is related to the sign of the 3-scalar of curvature $(3) R$, but, as was pointed out by Assad and Soares, the scalar of curvature is not a quantity that can in general be associated with the topological properties of the models.

A specially prominent characteristic of the previous solutions is that from them the corresponding vacuum solutions ($S=0$, as $\alpha \rightarrow 0$) can be recovered in a trivial way:

$$\varepsilon = +1: \quad x = (\tanh \frac{T}{2})^\beta, \quad y = (\tanh \frac{T}{2})^{-\beta} \sin T, \quad \beta^2 = 1; \quad (5a')$$

$$\varepsilon = -1: \quad x = (\tanh \frac{T}{2})^\beta, \quad y = (\tanh \frac{T}{2})^{-\beta} \sinh T, \quad \beta^2 = 1. \quad (5b')$$

These solutions were first obtained by Kantowski and Sachs and rediscovered by Vajk and Eltgroth, and Lorenz using different parametrizations. The present coincide with those of Lorenz. To get an idea of the influence of the scalar field on the models, eqs. (5a) and (5b) are rewritten in the form

$$\varepsilon = +1: \quad x = \frac{1}{\sqrt{f}} (\tanh \frac{T}{2})^\beta, \quad y = \frac{1}{\sqrt{f}} (\sin T) (\tanh \frac{T}{2})^{-\beta}$$

$$3\alpha^2 + \beta^2 = 1; \quad (5a'')$$

$$\varepsilon = -1: \quad x = \frac{1}{\sqrt{f}} (\tanh \frac{T}{2})^\beta, \quad y = \frac{1}{\sqrt{f}} (\sinh T) (\tanh \frac{T}{2})^{-\beta}$$

$$3\alpha^2 + \beta^2 = 1. \quad (5b'')$$
Then, one observes that the effect of the conformally invariant scalar field manifests itself through the factor $1/\sqrt{f}$ and the constraint $3\alpha^2 + \beta^2 = 1$. (It is worth noticing that $f \to 1$, as $\delta \to 0$.)

Calculating the curvature invariants for such metrics one finds that they diverge at $T = 2\pi \varepsilon (\varepsilon = 1)$ and $T = 0 (\varepsilon = -1)$, if the parameters $a$ and $\beta$ are such that

$$(\alpha, \beta) \in A, \ A = \{(a, b) \in R^2 | 3a^2 + b^2 = 1, \ 0 < a < \frac{1}{2}, \ -\frac{1}{2} < b < 0\}.$$  

(6)

On the other hand, the vacuum solutions $(5a')$ and $(5b')$ are singular at $T = 2\pi \varepsilon (\varepsilon = 1)$ and $T = 0 (\varepsilon = -1)$, respectively, if $\beta = 1$. So, if (6) holds, the conformally invariant scalar field cannot prevent the singularities. In the scalar-tensor theory proposed by Schmidt et al.\(^11\), a scalar field is used to avoid the problem of a singularity. Banerjee and Santos\(^14\) showed that, in general, this is not possible for a Friedmann-Robertson-Walker model. The scalar-tensor theory being dealt with here is a particular case of that presented by Schmidt et al.\(^2\).

The metric corresponding to solution (5c) is given by

$$ds^2 = \left[\frac{1}{T_1} - \alpha - \beta + T_1 + \alpha - \beta\right] \left(dx^2 + dy^2\right) - \left[\frac{1}{T_2} - \alpha + \beta + T_2 + \alpha + \beta\right] \left(dx^2 - dy^2\right) - \left[\frac{1}{T_3} - \alpha - \beta + T_3 + \alpha + \beta\right] \left(dx^2 \right).$$

(7)

When the conformally invariant scalar field is switched off ($\delta = 0$, as $\alpha \to 0$) this metric reduces to

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2$$

(8)

where the parameters $p_1$, $p_2$, and $p_3$ are given by\(^*$

$$p_1 = \frac{1-\beta}{2-\beta}, \ p_2 = p_3 = \frac{1-\beta}{2-\beta}$$

(9)

and satisfy the relationships

$$\sum_{\dot{\alpha}=1}^{3} p_{\dot{\alpha}}^2 = 1.$$  

(10)

\(^*\) From (5c) one has that $\beta^2 + 1$, as $\alpha \to 0$, so the possibility $\beta = 2$ is ruled out.
which is the Kasner empty universe\textsuperscript{15}.

The conformally flat metric is found to be a very special case when $a = \beta = \frac{1}{2}$. The metric (7) takes then the very simple form:

$$ds^2 = (1+2t)^2 (dx^2 - dy^2 - dz^2).$$ \hspace{1cm} (11)

The cosmological model represented by solution (7) is such that its proper volume is given by

$$V = \left[1+2t^2\alpha^2\right]^3 t^{2-3\alpha-\beta}.$$ \hspace{1cm} (12)

It is clear from the previous result that $V$ vanishes at $T=0$ if $(\alpha, \beta) \in \mathcal{B}, \mathcal{B} = \{(a, b) \in \mathbb{R}^2 | 3\alpha^2 + b^2 = 1, a \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, \frac{1}{\sqrt{3}}\right), b \in (-1, 0) \cup (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)\}$

Otherwise, if this relation is valid, $V$ becomes infinite as $T \to \infty$. Thus, the model starts from a point singularity and expands to an infinite volume\textsuperscript{*}. The Hubble parameters\textsuperscript{**} concerning the present model are given by

$$H_1 = \frac{x'}{x} = \frac{T^{\alpha+\beta-2} [\beta - \alpha + (\beta+\alpha) T^{2\alpha}]}{[1+T^{2\alpha}]^2}$$ \hspace{1cm} (14)

and it can be concluded that the model admits anisotropic expansions\textsuperscript{**} for $(\alpha, \beta) \in \mathcal{B}$, where $\mathcal{B}$ is given by (13)\textsuperscript{***}. Besides this, it is regular in the range $0 < T < \infty$.

\textsuperscript{*} The curvature invariants regarding solutions (5c) are found to diverge at $T=0$, if $(\alpha, \beta) \in \mathcal{C}, \mathcal{C} = \{(a, b) \in \mathbb{R}^2 | 3\alpha^2 + b^2 = 1, a \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, \frac{1}{\sqrt{3}}\right), b \in (-1, 0) \cup (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)\}$. Hence, $\mathcal{B} = \mathcal{C}$, and, as a consequence, the proper volume equals 0, where the curvature invariants of the model diverge.

\textsuperscript{**} These are kinematical parameters associated to the 4-velocities $\frac{dx}{dt}$ (in the coordinate system used in (1)).

\textsuperscript{***} The possibility $\alpha = \beta = \frac{1}{2}$ is forbidden by (13).
The model in view is consistent with the big-bang theory and is different from the solutions obtained by Accioly et al.\(^1\) and by Ram\(^3\).

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REFERENCES


Resumo

Se obtém soluções exatas das equações de Ei nstein- Campo- Estal Conformemente Invariante referentes a cosmologias de Kantowski-Sachs e Bianchi dos tipos I e III. A presença do campo escalar conformemente iriavariante é responsável por alguns aspectos interessantes das soluções. Encontra-se, em particular, que o modelo de Bianchi I é compatível com a teoria cosmológica do big-bang.