Lagrangian Solutions to the Three-Body Problem with Forces $r^P$ (p integer)+

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The exact solutions to the three-body problem in Celestial Mechanics, due to Lagrange (triangular solutions) and Euler (collinear solutions), are generalized to the case of forces $r^P$ ($p$ being an integer). The stability of the system is also investigated in a local sense (small variations about steady motion) for triangular and collinear solutions and conditions restricting the values of $p$ for which there are stable oscillatory modes of vibration are obtained. Furthermore, for the solutions under consideration, Bohr or Bohr-Sommerfeld quantization is performed and compared, for some cases of interest, with the WKB approximation, derived from an Hamiltonian of the system obtained by reducing it to a one-body problem under the action of a central force at the system's center of mass.

As soluções exatas do problema de três corpos na Mecânica Celestial, devidas a Lagrange (soluções triangulares) e Euler (soluções colineares), são generalizadas para o caso de forças $r^P$ ($p$ inteiro). A estabilidade do sistema é também investigada no sentido local (pequenas variações em torno da solução) para soluções triangulares e colineares, e são obtidas condições sobre os valores de $p$ para que correspondam a modos de vibração oscilatórios estáveis. Além disso, para as soluções em questão, as quantizações de Bohr e de Bohr-Sommerfeld são efetuadas e comparadas, para alguns casos de interesse, à aproximação WKB deduzida de um

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hamiltoniano do sistema obtido reduzindo-o a um problema de 1 corpo sob a ação de uma força central no centro de massa do sistema.

1. INTRODUCTION

It is well-known that the three-body problem in Celestial Mechanics cannot be solved, in general, in a closed form. The differential equations of motion of the system make the problem of the 18th order. This system of equations can be reduced to the 8th order by means of the integrals corresponding to conservation of the center of mass, energy and angular momentum. Lagrange proved that it is possible to reduce the problem to the 7th order and no further reduction has been obtained.

Lagrange also showed that, if the triangle formed by the three celestial bodies always remains similar to itself, then the three-body admits exact plane solutions and the bodies describe conic sections, similar to each other, with the center of mass being at one focus.

If the bodies are collinear, there are also exact solutions, which were first derived by Euler.

We recall that these classical solutions, due to Lagrange and Euler, have astronomical significance: a) the system composed by the sun, Jupiter and a Trojan asterid forms an equilateral triangle and was found to be a realization of the Lagrangian solution; b) the "gegenschein", that is, the patch of light (already observed by last century astronomers) due to a concentration of meteors on the line through the earth and the sun (collinear solution).

In the present note, it is shown in section 2, that Lagrange and Euler solutions can be generalized to the case of forces proportional to $r^{-p}$ (p integer).

In view of the above examples of realization of those solutions in nature (for Newtonian forces i.e. $p = 2$), it is reasonable to suppose that also in the case $p \neq 2$, these solutions may have a physical significance, even in the microscopical domain. This consideration moti-
vated us to study the stability (section 3) and the quantization (section 4) of those solutions, Section 5 is devoted to some final remarks.

2. GENERALIZED LAGRANGE AND EULER SOLUTIONS

We assume that the two-body forces between the bodies \( j \) and \( k \) are central and proportional to \( \frac{r^p}{r_{jk}^p} \), so that the total force acting on the body \( k \) is given by

\[
\sum_{j \neq k} C_{jk} \frac{r_{jk}^p}{r_{jk}^{p+1}}
\]

For \( p = 2 \) and \( C_{jk} = G \frac{m_j m_k}{r_{jk}} \), we have the classical case studied by Lagrange and Euler.

First, we note that it is easy to extend to any value of \( p \), \( p \neq 3 \), Carathéodory's proof\(^3\) that the plane formed by the bodies is fixed in space.

For each type of solution (triangular or rollinear) we shall consider two cases:

a) **Circular** (or "steady motion") : the mutual distances between the bodies are constants and the bodies rotate about their center of mass with an angular velocity \( \omega \).

b) **non-circular** (or "periodic") : the ratios between mutual distances remain constant and the orbits of the bodies vary according to the value of \( p \).

2.1. Lagrange type solutions

a) **Circular** case

The equations of motion, referred to the center of mass are given, in Cartesian coordinates by
\[ \ddot{z}_k = \sum_{j \neq k} m_j^{-1} C_{jk} \frac{x_j - z_j}{r_{jk}^{p+1}} \]  
\[ (j, k = 1, 2, 3). \]

\[ \ddot{y}_k = \sum_{j \neq k} m_j^{-1} C_{jk} \frac{y_j - y_k}{r_{jk}^{p+1}} \]  
\[ (2.1.1) \]

If \((\xi_k, \eta_k)\) rotates with angular velocity \(\omega\) with respect to the \((x_k, y_k)\) coordinates and if \(z_k = \xi_k + \eta_k\), then it follows from (2.1.1) that

\[ -z_k = \sum_{j \neq k} \omega^2 m_j^{-1} C_{jk} \frac{z_j - z_k}{r_{jk}^{p+1}} . \]  
\[ (2.1.2) \]

Defining now

\[ \rho_1 = \omega^2 C_{23} r_{23}^{-(p+1)} \]
\[ \rho_2 = \omega^2 C_{31} r_{31}^{-(p+1)} \]
\[ \rho_3 = \omega^2 C_{12} r_{12}^{-(p+1)} \]  
\[ (2.1.3) \]

we get, from (2.1.2), that

\[
\begin{align*}
(m_1 - \rho_2 - \rho_3) z_1 + \rho_3 z_2 + \rho_2 z_3 &= 0 \\
\rho_2 z_1 + \rho_1 z_2 + (m_3 - \rho_2 - \rho_1) z_3 &= 0 \\
m_1 z_1 + m_2 z_2 + m_3 z_3 &= 0
\end{align*}
\]  
\[ (2.1.3) \]

The last equation expresses that the center of mass is at rest. Eqs. (2.1.3) are consistent if

\[ \rho_1 = m_2 m_3 M^{-1} , \quad \rho_2 = m_1 m_3 M^{-1} , \quad \rho_3 = m_1 m_2 M^{-1} \]  
\[ (2.1.4) \]

where \(M\) is the total mass \(M = m_1 + m_2 + m_3\).

It follows that, for \(p \neq -1\),

\[ r_{jk} = \frac{C_{jk} M}{\omega^2 m_j m_k} \left( \frac{1}{p+1} \right) \]  
\[ j \neq k; \ j, k = 1, 2, 3 \]  
\[ (2.1.5) \]
For $p = -1$, that is, for harmonic forces we have

$$\rho_1 = \omega^2 C_{23}, \quad \rho_2 = \omega^2 C_{31}, \quad \rho_3 = \omega^2 C_{12}$$ (2.1.6)

and $C_{jk} = \omega^2 M^{-1} m_j m_k$: a solution exists for arbitrary $r_{jk}$.

We notice that, if $C_{jk} = C$ and $m_j = m$ or if $C_{jk} = G m_j m_k$, then from (2.1.5), one sees that $r_{12} = r_{23} = r_{31}$ and the bodies form an equilateral triangle.

b) Non-circular case

Let us suppose $C_{jk} = C$ and $m_j = m$ (equal masses case). Let $(x_3, y_3)$ be the coordinates at $t = t_0$, $r_{jk}$ the initial mutual distances and $(\xi_j, \eta_j)$ the coordinates at $t$ (with mutual distances $r_{jk}$, $\rho$ being the proportionality factor). We can write

$$\xi_j = \rho(x_3 \cos \theta - y_3 \sin \theta)$$
$$\eta_j = \rho(x_3 \sin \theta - y_3 \cos \theta)$$ (2.1.7)

where $\theta$ is the rotation angle.

Since

$$\xi_k = -\frac{C}{m} \sum_{j \neq k} \frac{\xi_j - \xi_j}{r_{jk}^{p+1}}$$
$$\eta_k = -\frac{C}{m} \sum_{j \neq k} \frac{\eta_j - \eta_j}{r_{jk}^{p+1}}$$ (2.1.8)

we are lead, from (2.1.7) and (2.1.8), to

$$\ddot{\rho} - \frac{y_k \psi}{x_k \rho} \cdot \frac{\psi^2}{\rho^3} = \frac{C}{m \rho} \sum_{j \neq k} \frac{x_j - x_k}{r_{jk}^{p+1}}$$
$$\ddot{\rho} + \frac{x_k \psi}{y_k \rho} \cdot \frac{\psi^2}{\rho^3} = \frac{C}{m \rho} \sum_{j \neq k} \frac{y_j - y_k}{r_{jk}^{p+1}}$$ (2.1.9)

$$\psi = \rho^2 \dot{\theta}$$.

Equations (2.1.9) are necessary conditions for the existence of solutions. They are consistent if
\[ \psi = 0 \] (2.1.10)

and

\[ \sum_{j \neq k} \frac{x_{j} - x_{j}^{d}}{r_{jk}^{p+1}} = \alpha x_{k} \] (2.1.11)

\[ \sum_{j \neq k} \frac{y_{j} - y_{j}^{d}}{r_{jk}^{p+1}} = \alpha y_{k} \]

Equations (2.1.11) are verified if

\[ r_{jk} = a \quad \text{and} \quad a = \frac{3}{\alpha^{1/p+1}} . \] (2.1.12)

And from (2.1.9) and (2.1.11), we get:

\[ \ddot{\phi} = \frac{\sigma_{0}^{2}}{\rho^{2}} \]

\[ \dot{\phi} = \frac{\rho^{2}}{\rho^{3}} \] (2.1.13)

The bodies, at any instant, form an equilateral triangle. The ratios between mutual distances remain equal and constant (for \( p < 2 \), see ref. 8).

2.2. Euler type solutions

a) Circular case

A similar treatment lead us to the equations

\[ -m_{1} \xi_{1} = \omega^{-2} \left[ \frac{C_{12}}{(\xi_{2} - \xi_{1})^{P}} + \frac{C_{31}}{(\xi_{3} - \xi_{1})^{P}} \right] \] (2.2.1)

\[ m_{3} \xi_{3} = \omega^{-2} \left[ \frac{C_{31}}{(\xi_{3} - \xi_{1})^{P}} + \frac{C_{23}}{(\xi_{3} - \xi_{2})^{P}} \right] \]

\[ m_{1} \xi_{1} + m_{2} \xi_{2} + m_{3} \xi_{3} = 0 \]

Let us define
Fig.1 - Three-body configuration corresponding to Euler solution with equal masses, in the circular case.

\[ a = \xi_2 - \xi_1 \]
\[ a\rho = \xi_3 - \xi_2 \]
\[ a(1+\rho) = \xi_3 - \xi_1 . \]  

We then obtain from (2.2.1), under the assumption that \( C_{jk} = C \) and \( m = m \)

\[ \frac{2 + \rho}{1 + 2\rho} = \frac{1 + (1 + \rho)^{-p}}{(1 + \rho)^{-p} + \rho^{-p}} \]  

(2.2.3)

Notice that \( \rho = 1 \) verifies (2.2.3) for any \( p \) and corresponds to

\[ \xi_2 - \xi_1 = a, \quad \xi_3 - \xi_2 = a, \quad \xi_3 - \xi_1 = 2a . \]

Thus, if \( C_{jk} = C \) and \( m = m \), there are solutions for any \( p \), corresponding to the situation of Fig.1. If \( p = -1 \), (2.2.3) becomes an identity and so one has a solution for arbitrary \( \xi_j \).

b) Non-circular case

As in section 2.1 b (non-circular Lagrange solutions), we can derive

\[ \sum_{j\neq k} \frac{x_k - x_j}{r_{jk}^{p+1}} = a x_k \]  

(2.2.4)

\[ \sum_{j\neq k} \frac{y_k - y_j}{r_{jk}^{p+1}} = a y_k \]
In this case, one must have

\[ \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} \]

and (2.2.4) is verified if

\[ r_{12} = r_{23} = a; \quad r_{13} = 2a; \quad a = \frac{1+p}{c^p+1}. \]

Furthermore, we have

\[ \ddot{\beta} = \frac{c_0^2}{\rho^3} = -\frac{(1+2^{-p})}{m\alpha^{p+1}} \left( \frac{1}{\rho^p} \right) \]

(2.2.5)

\[ \dot{\theta} = c_0 \rho^{-2}. \]

If \( p = -1 \), \( a = 3\alpha^{-2} \), which coincides with the value of \( a \) for the triangular case (2.1.12).

In the non-circular case (triangular or collinear) it is possible, for \( p = -5, -3, -1, 0, 2, 3, 4, 5 \) and 7, to solve (by circular or elliptic functions) the integral that gives the orbits of the bodies. If \( p = 2 \), the orbits are conics with the center of mass at one focus. For \( p = -1 \), the orbits are still conics but the center is at the center of mass.

3. STABILITY OF THE SOLUTIONS

3.1. Stability of Lagrange type solutions

For \( C_{jk} = G m_j m_k \), we shall consider small vibrations about steady motion, in the circular case.

Let us take the origin at \( m_3 \) and, as coordinates, the distances \( r_{13}, r_{23} \) and the angles \( \alpha_1 \) and \( \alpha_2 \), as in Fig. 2. As auxiliary variables we have: \( r_{12}, \theta_1, \theta_2, \) and \( \theta_3 \). The ignorable coordinate does not appear explicitly.
Writing the equations of motion in terms of those coordinates and taking solutions near Lagrange's steady motion, we can derive a secular determinant from which results the condition (p ≠ 3):

$$K^2 \left( M + \frac{K^2}{3-p} \right) \left( K^4 + (3-p)MK^2 + \frac{3}{4} \left( m_1 m_2 + m_1 m_3 + m_2 m_3 \right) (p+1)^2 \right)$$

(3.1.1)

where \( K \) stands for the frequency of oscillation and \( M \) is the total mass.

The roots of (3.1.1) are

(a) \( K^2 = 0 \), which corresponds to the additional ignorable coordinate.

(b) \( K^2 = -(3-p) \), for which there are stable oscillatory modes if

$$p < 3$$

(3.1.2)

(c) \( K^2 = -(3-p) \frac{M}{2} \pm \left( (3-p)^2 M^2 - 3(p+1)^2 (m_1 m_2 + m_2 m_3 + m_1 m_3) \right)^{1/2} \),

which corresponds to stable modes if

$$M^2 (3-p)^2 > 3(p+1)^2 \left( m_1 m_2 + m_2 m_3 + m_1 m_3 \right).$$

(3.1.3)

For the case \( p=3 \), we verify that there are no stable modes.
Thus, (3.1.2) and (3.1.3) are the conditions of stability for Lagrange type circular solutions.

3.2. Stability of Euler type solutions

Let us suppose $C_{jk} = m^2G$ and take the coordinates $r_{13}, r_{23}, \alpha_1$ and $a$ as in Lagrange's case. The Euler type solutions correspond to the values

$$r_{13} = a, \quad r_{23} = 2a, \quad \alpha_1 = a = \omega t.$$

As in subsection 3.1, we look for solutions near the steady motion. From the equations of motion, we obtain, to the first order, four equations in the small variables, which lead to a secular determinant whose roots are given by

(a) $K = 0$.

(b) $K^2 = (p-3)\omega^2$.

(c) $K^2 = -\frac{A}{2} \pm \frac{1}{2} (A^2 - 4B)^{1/2}$

where

$$A = \frac{\omega^2}{1 + 2^{-p}} (5 - 3p + 2^{-p+1})$$

$$B = -\frac{2\omega^2}{1 + 2^{-p}} \left[ 1 + 2^{-(p+1)} + 3p(1 - 2^{-(p+1)}) - 2^{-(2p+1)} \right]$$

Thus, the conditions for stable modes are

from b):

$$p < 3 ; \quad (3.2.1)$$

from c):

$$p < -3 . \quad (3.2.2)$$

We conclude that only if $p < -3$, there is stability for Euler type circular solutions, in the equal masses case.
4. SEMI-CLASSICAL QUANTIZATION OF THE SOLUTIONS

In this section, we perform the quantization of the generalized solutions, applying first Bohr quantization to the simplest case of circular solutions and Bohr-Sommerfeld quantization for the non-circular ones, for both Lagrangian and Eulerian type cases.

4.1. Bohr's quantization

a) Lagrange circular solutions

Let us suppose $C_{jk} = C$ and $m_j = m$. The total energy $E$ is given, in this case, for $p \neq 1$, by

$$E = \frac{1}{2} \frac{p-3}{p-1} m \omega^2 a^2$$  \hspace{1cm} (4.1.1)

with

$$\omega^2 = \frac{3C}{ma^{p+1}}$$  \hspace{1cm} (4.1.2)

The angular momentum w.r.t the center of mass is

$$L = ma \omega^2 = n\hbar, \; n = 1, 2, \ldots$$  \hspace{1cm} (4.1.3)

It follows, for $p \neq 1$ and $p \neq 3$, that

$$E_n = \frac{1}{2} \frac{p-3}{p-1} (3C)^{3-p} \left( \frac{n^2 \hbar^2}{m} \right)^{\frac{1-p}{3-p}}$$  \hspace{1cm} (4.1.4)

If $p = 3$, we have $E = 0$. For $p = 1$, corresponding to the logarithmic two-body potential

$$V(p, n) = C \ln \frac{n_{jk}}{r_0}, \; \text{we get}$$

$$E_n = \frac{3C}{2} \left[ 1 + \ln \left( \frac{n^2 \hbar^2}{3mCn_0^2} \right) \right].$$  \hspace{1cm} (4.1.5)
b) Euler type circular solutions

For \( p \neq 1 \) and \( p \neq 3 \), we obtain in a similar way, the following expression for the quantized energies (equal masses case).

\[
E_n = \frac{2(1-p)}{p-3} \frac{p-3}{p-1} \left(1 + \frac{2}{(1+2^{-p})C}\right)^{\frac{2}{3-p}} \left(\frac{n^2h^2}{m}\right)^{\frac{p-1}{p-3}}
\]  

(4.1.6)

Again, if \( p=3 \), we have \( E=0 \) and for \( p=1 \), we are lead to

\[
E_n = \frac{3C}{2} \left[1 + \ln \left(\frac{n^2h^2}{6mC_a^2}\right)\right].
\]  

(4.1.7)

We see, from (4.1.4) and (4.1.6), that for \( p=2 \) one gets a Balmer-like spectrum. Notice also that, if \( p=-1 \) (harmonic forces), (4.1.4) and (4.1.6) give rise to identical expressions.

4.2. Bohr-Sommerfeld quantization of non-circular solutions

An important feature of the non-circular solutions is that the corresponding equations of motion can be derived from an Hamiltonian \( H(q,p) \), where the canonical variable is \( q = p \) (\( p \) is the already defined ratio of the mutual distances between the bodies) and the conjugate momentum \( p = ma^2 \). Indeed, from the expressions for the kinetic and potential energies of the system, we obtain

\[
H(q,p) = \frac{1}{2} \frac{p^2}{ma^2} + \frac{1}{2} m a^2 \alpha^2 \rho^{-2}
\]

\[
= \begin{cases} \frac{\alpha C a^2}{p-1} \rho^{1-p}, & \text{if } p \neq 1 \\ -\alpha C a^2 \ln \rho, & \text{if } p = 1 \end{cases}
\]  

(4.2.1)

with

\[
\alpha = \begin{cases} a^{-(p+1)}, & \text{(triangular case)} \\ (1 + 2^{-p}) a^{-(p+1)}, & \text{(collinear case)} \end{cases}
\]
Therefore, for those solutions, the system of three bodies is equivalent to one body under the action of a central force at the system's center of mass. In order to accomplish the Bohr-Sommerfeld quantization, we introduce the action variable $J_\rho = \int p_\rho \, d_\rho$, where $p_\rho$ is obtained by means of (4.2.1) through the equation $H(p_\rho, p_\rho) = E$. The action integral can then be solved by the method of residues, for the cases $\rho = 2$, $\rho = -1$ and $\rho = 3$.

a) Lagrange type solutions

We quote the following results: i) $\rho = 2$, 

$$E_n = - \frac{9mc^2}{2\hbar^2} \frac{1}{n^2}, \quad n = 1, 2, \ldots \quad (4.2.2)$$

a result which coincides with that obtained from (4.1.4) for $\rho = 2$;

ii) for $\rho = -1$, 

$$E_n = \left( \frac{3C}{m} \right)^{1/2} \hbar n, \quad n = 1, 2, \ldots \quad (4.2.3)$$

Again, the result coincides with (4.1.4) for this value of $\rho$.

b) Euler type solutions

We obtain the following results:

i) for $\rho = 2$, 

$$E_n = - \frac{25}{16} \frac{mc^2}{\hbar^2} \frac{1}{2n^2} . \quad (4.2.4)$$

ii) for $\rho = -1$, 

$$E_n = \left( \frac{3C}{m} \right)^{1/2} \hbar n . \quad (4.2.5)$$
4.3. The WKB approximation

Starting from the Hamiltonian (4.2.1), which describes the solutions under consideration, we can write the corresponding Schrödinger equation, whose eigenvalues are the quantized energies of the system. For instance, for the Newtonian case \((p=2)\), in the unequal masses case, one can readily derive, from well-known quantum-mechanical results, the expression:

\[
E_n = - \frac{G^2 (m_1 m_2 + m_2 m_3 + m_3 m_1)^3}{(m_1 + m_2 + m_3) \hbar^2} \frac{1}{2n^2}, \quad n = 1, 2, \ldots (4.3.1)
\]
valid for the Lagrangian solution.

The same formula can be obtained by means of Bohr or Bohr-Sommerfeld quantization conditions.

We may apply the WKB approximation to get energy expressions in some significant cases (for instance, the logarithmic potential case \(p=1\), and the linear potential case \(p=0\)).

With the Hamiltonian (4.1.1), we have

\[
J_\rho = \oint p_\rho \, d\rho = a \oint \left[2m(E-V_{\text{eff}})\right]^{1/2} \, d\rho
\]

(4.3.2)

where

\[
V_{\text{eff}}(\rho) = \frac{ma^2 c^2}{2\rho^2} - \begin{cases}
\frac{\alpha c^2}{(p-1)\rho^{p-1}} \\
-\alpha c^2 \ln \rho.
\end{cases}
\]

(4.3.3)

In the WKB approximation for the three-dimensional problem, we have

\[
J_\rho = 2\pi \hbar \left(n - \frac{1}{4}\right), \quad n = 1, 2, \ldots (4.3.4)
\]

Considering the case of S-waves \((c_0=0)\), the results for Lagrange type solutions are:
For \( n = 1 \), (4.3.6) gives
\[
E_1 = 2.3202 \left( \frac{9C^2\bar{R}^2}{2m} \right)^{1/3}.
\]
(notice that the exact factor is 2.3381).

For the Euler type solutions, we obtain:

i) \( p = 1 \), \( V_{\text{eff}} = \frac{3C}{2} \ln \frac{r}{r_0} \),
\[
E_n = \frac{3C}{2} \ln \left( n - \frac{1}{4} \right) \left( \frac{2\pi}{mc} \right)^{1/2} \frac{\bar{R}}{r_0}.
\]  
(4.3.7)

ii) \( p = 0 \), \( V_{\text{eff}} = 2Cr \);
\[
E_n = \frac{1}{2} \left[ 6 \frac{C\bar{R}n}{\sqrt{m}} \left( n - \frac{1}{4} \right) \right]^{2/3}.
\]  
(4.3.8)

The above results are to be compared with those obtained in section (4.1) for \( p = 0 \) and \( p = 1 \).

5. FINAL REMARKS

We have shown that it is possible to generalize Lagrange and Euler type exact solutions to the three-body problem for forces proportional to \( r^{-p} \) \((p\ \text{integer})\).

The conditions of stability of the circular solutions have also been established for arbitrary values of \( p \).
For \( p = -1 \) (harmonic forces), there are triangular and collinear solutions for arbitrary \( \epsilon_{jk} \). This occurs because, as is well known, the problem is always separable. Equilateral triangular solutions are always stable, even if the masses are different. Collinear solutions with equal masses are not stable. The quantized energies (equal masses) are identical in Lagrange's and Euler's case (circular and non-circular, see eqs. 4.2.3 and 4.2.5).

For Newtonian forces \( (p=2) \) with equal masses, the solutions are not stable (see eqs. 3.1.3 and 3.2.2). In both cases, a Balmer-like formula is obtained, even if the masses are distinct (see for instance, eq. 4.2.1).

We notice that it was shown\(^9\) that particular solutions also exist for the case of four-bodies with forces proportional to \( r^p \), which are analogous to the Lagrange and Euler solutions here discussed for the 3-body problem.

Finally, we think it would be interesting to speculate about the possible existence of microscopical systems realizing the Lagrangian or Eulerian solutions in that domain. However, no attempt in this direction was made here.

REFERENCES AND NOTES

2. A proof that the movement in this case occurs in a fixed plane was given by Lagrange, and in a more elementary form by Carathéodory (see ref. 3).
6. Some authors refer to both of them as Lagrange solutions. Following
H. Pollard (ref.7), we prefer to distinguish between Euler's (collinear) and Lagrange's (triangular) case.


