Revista Brasileira de Física, Vol. 7, Nº *3,* 1977

Application of Analytic Regularization to the Casimir Forces*

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Recebido em 26 de Outubro de 1977

Analytic regularization for the Casimir Effect for rectangular systems in one-, two- and three-dimensions, as well as for parallel conducting plates, is discussed. We consider the analytic regularization by employing the Riemann zeta function as well as the zeta functions introduced by Epstein. The forces, in this case, come out automatically finite, i. e., no subtractions are needed. We show that the analytic continuation, in the number of imaginary time dimensions, corresponds to introducing generalized zeta functions for the **zero point** energy.

Discutimos a regularização analítica para o Efeito Casimir em sistemas retangulares a uma, duas e três dimensões, assim como para placas condutoras e paralelas. No processo de regularização analìtica fazemos uso da função zeta de Riemann assim como das funções zeta introduzidas por Epstein. As forças, neste caso, são automaticamente finitas, sem a necessidade de subtrações. Mostramos que a continuação analitica, no número de dimensões temporais imaginárias, corresponde à introdução de funções zeta generalizadas para a energia do ponto zero.

* Work supported by FINEP, Rio de Janeiro, under contract 356/CT.
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1. INTRODUCTION

Casimir[1,2] has studied the effect of the zero point energy of the electromagnetic field on two parallel conducting plates. His calculation presents infinities, and in order to obtain a finite result, one subtracts from the energy density the corresponding contribution from the whole space (the plates being separated by an infinite distance).

The same procedure has been extended by Lukosz[3] to rectangular systems.

The purpose of this paper is to study how to obtain regularized quantities by a procedure similar to the one presented by Gelfand and Shilov in their book[4]. Such methods of analytic regularization were extended to quantum field theory by Bollini, Giambiagi and Gonzáles Dominguez[5].

We make the remark that the exponential cut-offs used in Refs.1,2,3 can be interpreted as analytic regulators. In the case of a one-dimensional box, we also introduce in this paper an analytic parameter which allows one to calculate the Casimir forces in terms of the Riemann zeta function. For rectangular boxes of higher dimensions we also consider, after discussing the results with the help of exponential regulators, the analytic regularization by means of generalized zeta functions first introduced by Epstein[7].

In these rectangular syçtems, with the help of these zeta functions, the quantities which measure the Casimir forces are automatically finite, that is, no subtractions are needed (in general, the infinities are due to pole terms).

In Section 2, we describe the Casimir effect in a one-dimensional box, and in Section 3 for the system of two parallel conducting plates. We discuss the regularization by means of exponential regulators, which exibit poles, and which in analogy to what happens in other situations of quantum field teory, have to be subtracted[5]. We also consider a regulating parameter which naturally gives rise to the Riemann zeta function, and which when continued analitically to the physical region produces automatically finite results for the forces.

In Sections 4 and 5, we consider in a similar way boxes in two- and three-dimensions. In these cases, generalizations of zeta functions, first considered by Epstein[7], do appear.

In Section 6, we study the connection of our approach with the analytic regularization of the Green function. In particular, as we have given boundary conditions in space, we make the analytic continuation in the number of imaginary time dimensions. In this way, the zero point energy is naturally expressed in terms of generalized zeta functions.

In the Appendices, we discuss some properties, known in the mathematical literature, and which might be useful to the eventual reader. So, in Appendix A, we show that Riemann's zeta function $\zeta(s)$, for Re $s<1$, is a finite part (in the sense of Hadamard) of an integral, and that it satisfies the correct functional relation; therefore, this finite part is the right analytical continuation for Re $s<1$. In Appendix B, we give a derivation of the functional equation of the zeta function introduced by Epstein[7], and which allows one to perform the analytical continuation.

In order to make this article rather self-contained, much of the arguments found in other papers, especially of Ref.3, are reproduced here for the convenience of the prospective reader.

In the examples treated in this paper, that is, the Casimir effect in rectangular systems, there is no difficulty in obtaining the finite parts by using the exponential regulator and separating the poles, or by introducing a parameter which allows one to consider generalized zeta functions. In these examples, there is no special advantage in using the analytical regularization with the zeta function technique instead of the exponential one. But there are problems, like the Casimir effect in a spherical conducting shell, where the final result is not mathematically well settled[9,10] when the exponential regulator is used. The reason is that, for this problem, the separation of the poles (or other type of singularities) is not very clean. In this case, the analytical regularization via the zeta function technique can be useful.

2. ONE-DIMENSIONAL BOX

Let us consider the simple case of a massless scalar field in a one-dimensional box of length L. With the boundary condition that the field is zero at the ends, the eigenfrequencies are given by

$$\omega_n = \frac{\pi c}{L} n \, , \quad n = 1,2,\ldots, \tag{1}$$

where c is the wave velocity. The zero point energy is

$$E_0 = \frac{1}{2}\hbar \sum_{n=1}^{\infty} \omega_n = \frac{1}{2}\frac{\pi\hbar c}{L} \sum_{n=1}^{\infty} n \, , \tag{2}$$

which is infinite. In order to obtain meaningful results, people have introduced a frequency cut-off. We can, for instance, consider[2]:

$$E_0 = \frac{1}{2}\hbar \lim_{\alpha\to 0_+} \sum_{n=1}^{\infty} \omega_n e^{-\alpha\omega_n} = -\frac{\hbar}{2} \lim_{\alpha\to 0_+} \frac{\partial}{\partial\alpha} \sum_{n=1}^{\infty} e^{-\alpha\omega_n} \tag{3}$$

or

$$E_0 = -\frac{\hbar}{2} \lim_{\alpha\to 0_+} \frac{\partial}{\partial\alpha} \sum_{n=1}^{\infty} e^{-\alpha\pi c n/L} = \lim_{\alpha\to 0_+} \left\{ -\frac{\hbar L}{2\pi c}\frac{B_0}{\alpha^2} - \frac{\hbar\pi c}{4L} B_2 + O(\alpha) \right\} \tag{4}$$

where use was made of the formula

$$\frac{y}{e^y - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} y^\nu \, , \tag{5}$$

the B_ν being Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, etc.

We see in Eq. (3) that the first term gives a divergent contribution for E_0. Dividing this term by L, we see that it gives a contribution to the energy density which is independent of L, and therefore represents the energy density of an infinite box. The usual prescription is to subtract this infinite contribution, and what remains is the physical zero point energy in the box of dimension L, i.e.,

$$E_0^R = -\frac{\pi\hbar c}{4L} B_2 \quad . \tag{6}$$

Other kinds of frequency cut-off have been used[3], with identical prescriptions, in order to eliminate infinities, the same final finite results being obtained.

Using

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \quad , \tag{7}$$

where $\zeta(s)$ is the Riemann zeta function, Eq. (6) can be written as

$$E_0^R = \frac{\pi\hbar c}{2L} \quad \zeta(-1) \quad . \tag{8}$$

Let us now remark that the identification given by Eq. (3) corresponds to an analytic regularization with parameter $\alpha>0$, and the prescription for obtaining the finite part of the lefthand side of Eq. (4) is to subtract the corresponding poles in a. This situation is completely analogous to those which occur in quantum field theory[5].

Instead of using a regularization of the type of Eq.(3), we can also use

$$\sum_{n=1}^{\infty} \omega_n = \lim_{s\to -1} \sum_{n=1}^{\infty} \omega_n^{-s} = \frac{\pi c}{L} \lim_{s\to -1} \sum_{n=1}^{\infty} n^{-s} \quad . \tag{9}$$

For Re $s>1$, the righthand side of Eq.(9) defines the Riemann reta function $\zeta(s)$, which can be continued analitically for Re $s<1$ (Ref.6). This can be done for instance by considering the finite part *ã la* Hadamard[4] (cf. Appendix A).

By this procedure, Eq.(2) writes as Eq.(8), and the zero point energy of an infinite space ($L\to\infty$) is zero in this case. By using this type of analitic regularization, we obtain automatically finite results, no subtractions being needed.

3. TWO CONDUCTING PARALLEL PLATES

The zero point energy is given by

$$E_0 = \hbar \sum_{\vec{k},n} \omega(\vec{k},n) \quad ,$$

with

$$\omega(\vec{k},n) = \sqrt{k_x^2+k_y^2+\frac{n^2\pi^2}{d^2}} \quad ,$$

where we have put c = 1; d is the distance between the plates.

We have

$$\sum_{\vec{k},n} \omega(\vec{k},n) = \frac{S}{\pi^2} \sum_{n=0}^{\infty} \int_0^{\infty}\int_0^{\infty} dk_x dk_y \sqrt{k_x^2+k_y^2+\frac{n^2\pi^2}{d^2}} \quad , \tag{10}$$

S denoting the surface of the plates. For $n=0$, the above expression should be multiplied by 1/2. Since, for $n=0$, the contribution for E_0 is independent of d, we can subtract it (because it does not contribute to the forces) and then restrict the sum in Eq. (10) to $n=1,2,\ldots$. Introducing into Eq. (10) the exponential regulator $\exp\{-\alpha(k_x^2+k_y^2+n^2\pi^2/d^2)^{1/2}\}$, we shall obtain[2] a fourth order pole in a which will give an energy density independent of d. By the same procedure as before, we subtract it. It also appears a third order pole which is independent of d, and therefore it has also to be subtracted since it does not contribute to the forces. Therefore, we are left with the regularized expression

$$\left(\sum_{\vec{k},n} \omega(\vec{k},n)\right)^{R} = \frac{\pi^2 S}{d^3}\,\frac{1}{4!}\,B_4 \quad . \tag{11}$$

In analogy to Eq. (9), we write

$$\left(\sum_{\vec{k},n} \omega(\vec{k},n) \right)^{R} = \lim_{s\to-1} \sum_{\vec{k},n} \left[\omega(\vec{k},n)\right]^{-s} .$$

We have:

$$I_s = \sum_{\vec{k},n} \left[\omega(\vec{k},n)\right]^{-s} = \frac{S}{\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty}\int_0^{\infty} dk_x dk_y \left[\sqrt{k_x^2 + k_y^2 + n^2\pi^2/d^2}\,\right]^{-s} ,$$

where again the contribution from $n=0$ was disregarded, since it is independent of d. We remark. that, in this case, this term is proportional to $\int_0^{\infty} \rho^{-s+1}\, d\rho$, whose analytic regularization vanishes (cf.Ref.4, p.70).

It is easy to show that

$$I_s = \frac{S}{4\pi} \cdot \sum_{n=1}^{\infty} \left(\frac{n\pi}{d}\right)^{-s+2} \int_0^{\infty} dy\,(y+1)^{-s/2}$$

Now,

$$\int_0^{\infty} dy\,(y+1)^{-s/2} = B(1, \tfrac{s}{2} -1) ,$$

which is valid for Re $s>2$, $B(p,q)$ being the Euler beta function. Therefore,

$$I_s = \frac{S}{4\pi} B(1, \tfrac{s}{2} -1) \sum_{n=1}^{\infty} \left(\frac{n\pi}{d}\right)^{-s+2} .$$

The righthand side of this expression is convergent for Re $s>3$, and defines the function

$$I_s = \frac{S}{4\pi} \left(\frac{\pi}{d}\right)^{-s+2} B(1,\tfrac{s}{2} - 1)\, \zeta(s-2) .$$

This expression can be continued analitically for Re $s<3$. In particular, for $s=-1$, we have:

$$I_{-1} = -\frac{\pi^2}{6} \frac{S}{d^3} \zeta(-3) ,$$

which is exactly Eq.(11) after use of Eq.(7).

4. TWO DIMENSIONAL RECTANGULAR BOX

Let us consider a scalar field in a two-dimensional rectangular box with the boundary condition that the field is zero at the walls.

The solutions are then of the type

$$\phi(x,y) = A \sin\frac{\pi n_1 x}{L_1} \sin\frac{\pi n_2 y}{L_2},$$

with $n_i = 1,2,\ldots$ $(i = 1,2)$. The eigenfrequencieç are

$$\omega_{n_1 n_2} = \pi\left[(n_1/L_1)^2 + (n_2/L_2)^2\right]^{1/2},$$

with c = 1. The zero point energy is

$$E_0(L_1,L_2) = \frac{1}{2}\hbar \sum_{n_1,n_2=1}^{\infty} \omega_{n_1 n_2} = \frac{\pi\hbar}{2} \sum_{n_1,n_2=1}^{\infty} \left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}. \quad (12)$$

Let us now go to the exponential regularization:

$$\sum_{n_1,n_2=1}^{\infty} \omega_{n_1 n_2} = -\lim_{\alpha\to 0_+} \frac{\partial}{\partial\alpha} \sum_{n_1,n_2=1}^{\infty} \exp(-\alpha\omega_{n_1 n_2}) \quad (13)$$

Now we have:

$$\sum_{n_1,n_2=1}^{\infty} \exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\}$$

$$= {\sum_{n_1,n_2=0}^{\infty}}' \exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\} - \sum_{n_1=1}^{\infty} \exp\{-\alpha(n_1/L_1)\} - \sum_{n_2=1}^{\infty} \exp\{-\alpha(n_2/L_2)\}$$

$$= \frac{1}{4} \sum_{n_1,n_2=-\infty}^{\infty}{}' \exp\{-\alpha[(n_1/L_1)^2+(n_2/L_2)^2]^{1/2}\}$$

$$- \frac{1}{2} \sum_{n_1=1}^{\infty} \exp\{-\alpha(n_1/L_1)\} - \frac{1}{2} \sum_{n_2=1}^{\infty} \exp\{-\alpha(n_2/L_2)\} - \frac{1}{4}, \qquad (14)$$

where the prime means that the case where $n_1=n_2=0$ is excluded. The Euler-Maclaurin sum formula gives

$$\sum_{n_i=1}^{\infty} \exp(-\alpha n_i/L_i) = \int_0^{\infty} \exp(-\alpha x/L_i)\, dx - \frac{1}{2} + \frac{\alpha}{12L_i} + 0(\alpha^2), \qquad (15)$$

while with the Poisson summation formula one has

$$\sum_{n_1,n_2=-\infty}^{\infty} \exp\{-\alpha[(n_1/L_1)^2+(n_2/L_2)^2]^{1/2}\}$$

$$= \sum_{m_1,m_2=-\infty}^{\infty} \iint_{-\infty}^{+\infty} \exp\{-\alpha[(x/L_1)^2+(y/L_2)^2]^{1/2} + 2\pi i(m_1x+m_2y)\}\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \exp\{-\alpha[(x/L_1)^2+(y/L_2)^2]^{1/2}\}\, dx\, dy +$$

$$\sum_{m_1,m_2=-\infty}^{\infty}{}' \iint_{-\infty}^{\infty} \exp\{-\alpha[(x/L_1)^2+(y/L_2)^2]^{1/2} + 2\pi i(m_1x+m_2y)\}dxdy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\alpha t}\, t dt d\theta + \sum_{m_1,m_2=-\infty}^{\infty}{}' \int_0^{\infty}\int_0^{2\pi} \exp\{-\alpha t+2\pi i \ell t\cos\theta\} t\, dt\, d\theta, \qquad (16)$$

where in the very last sum the terrn $m_1=m_2=0$ is excluded. Note that we have made use of the vectors $\vec{t} = (x/L_1, y/L_2)$ and $\vec{\ell} = (m_1L_1, m_2L_2)$.

Eq.(16) can be written as

$$\sum_{n_1,n_2=-\infty}^{\infty} \exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\} = 2\pi L_1 L_2 \int_0^\infty e^{-\alpha t}\, t dt \; +$$

$$\sum_{m_1,m_2=-\infty}^{\infty}{}' \frac{2\pi\alpha L_1 L_2}{(\alpha^2+4\pi^2\ell^2)^{3/2}} \; . \tag{17}$$

Introducing Eqs.(15) and (17) into Eq.(14), it then follows that

$$\sum_{n_1,n_2=1}^{\infty} \exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2} =$$

$$\frac{\pi}{2} L_1 L_2 \int_0^\infty t\, e^{-\alpha t}\, dt - \frac{1}{2}(L_1+L_2)\int_0^\infty e^{-\alpha t}\, dt + \frac{1}{4} \; -$$

$$\frac{\alpha}{24}\left(\frac{1}{L_1}+\frac{1}{L_2}\right) + \frac{\pi}{2}\alpha L_1 L_2 \sum_{m_1,m_2=-\infty}^{\infty}{}' \frac{1}{(\alpha^2+4\pi^2\ell^2)^{3/2}} + 0(\alpha^2) \, . \tag{18}$$

The first two terms give pole terms of order two and one, respectively. Introducing Eq.(18) into Eq.(13) and subtracting the pole terms in a, we obtain the following regularized expression:

$$\left\{\sum_{n_1,n_2=1}^{\infty} \left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\right\}^R =$$

$$\frac{1}{24}\left(\frac{1}{L_1}+\frac{1}{L_2}\right) - \frac{L_1L_2}{16\pi^2}\sum_{m_1,m_2=-\infty}^{\infty}{}' (m_1^2L_1^2+m_2^2L_2^2)^{-3/2} \; . \tag{19}$$

Let us now make the difference between the energy densities of a box, area L_1L_2, and another of area $L_1'L_2'$ with $L_1', L_2' \to \infty$. We have then, from Eq.(12),

$\Delta u(L_1,L_2) =$

$$-\frac{\pi\hbar}{2L_1L_2}\lim_{\alpha\to 0_+}\frac{\partial}{\partial\alpha}\left\{\sum_{n_1,n_2=1}^{\infty}\exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\} - \right.$$

$$\left.\iint_{-\infty}^{\infty}{}^{*}\exp\{-\alpha\left[(x/L_1)^2+(y/L_2)^2\right]^{1/2}\}\,dxdy\right. ,$$

where the asterisk means that terms with x = 0 and y = 0 must be excluded. That is:

$$\iint^{\infty}{}^{*}\exp\{-\alpha\left[(x/L_1)^2+(y/L_2)^2\right]^{1/2}\,dxdy =$$

$$\iint_{0}^{\infty}\exp\{-\alpha\left[(x/L_1)^2+(y/L_2)^2\right]^{1/2}\}\left[1-\delta(x)-\delta(y)\right]dx\,dy ,$$

which gives exactly the result given by Eq.(19) multiplied by $\pi\hbar/2L_1L_2$. Let us consider the identity

$$\sum_{n_1,n_2=1}^{\infty}\left[\left(\frac{n_1}{L_1}\right)^2+\left(\frac{n_2}{L_2}\right)^2\right]^{-s} =$$

$$\frac{1}{4}\sum_{n_1,n_2=-\infty}^{\infty}{}'\left[\left(\frac{n_1}{L_1}\right)^2+\left(\frac{n_2}{L_2}\right)^2\right]^{-s}-\frac{1}{2}\sum_{n_1=1}^{\infty}\left(\frac{n_1}{L_1}\right)^{-2s}-\frac{1}{2}\sum_{n_2=1}^{\infty}\left(\frac{n_2}{L_2}\right)^{-2s} ,$$

which for Re $s>1$ can be substituted by

$$\sum_{n_1,n_2=1}^{\infty}\left[\left(\frac{n_1}{L_1}\right)^2+\left(\frac{n_2}{L_2}\right)^2\right]^{-s}=\frac{1}{4}A(L_1,L_2;2s)-\frac{1}{2}(L_1^{2s}+L_2^{2s})\ \zeta(2s),$$

where $A(L_1,L_2;2s)$ is an Epstein zeta function[7] defined in Appendix B. Using the functíonal equation (B.4), it follows for $s=-1/2$ that

$$\left\{\sum_{n_1,n_2=1}^{\infty}\left[(n_1/L_1)^2+(n_2/L_2)^2\right]^{1/2}\right\}^R =$$

$$\frac{1}{24}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)-\frac{L_1L_2}{16\pi^2}\sum_{m_1,m_2=-\infty}^{\infty}{}' (m_1^2L_1^2+m_2^2L_2^2)^{-3/2} \ ,$$

which is exactly Eq.(19).

5. THREE-DIMENSIONAL BOX

Let us now consider a scalar field in a three-dimensional box of volume L_1 L, L,, with the boundary condition that the field vanishes at the walls.

The solutions are of the form:

$$\phi(x,y,z) = A \sin\frac{\pi n_1 x}{L_1}\sin\frac{\pi n_2 y}{L_2}\sin\frac{\pi n_3 z}{L_3} \ ,$$

with $n_i = 1,2,3$ $(i = 1,2,3)$. The eigenfrequencieç are

$$\omega_{n_1,n_2,n_3} = \pi\left[(n_1/L_1)^2+(n_2/L_2)^2+(n_3/L_3)^2\right]^{1/2} .$$

Taking the exponential regularization, i.e.,

$$\sum_{n_1,n_2,n_3=1}^{\infty}\left[(n_1/L_1)^2+(n_2/L_2)^2+(n_3/L_3)^2\right]^{1/2} =$$

$$-\lim_{\alpha\to 0_+}\frac{\partial}{\partial\alpha}\sum_{n_1,n_2,n_3=1}^{\infty}\exp\{-\alpha\left[(n_1/L_1)^2+(n_2/L_2)^2+(n_3/L_3)^2\right]^{1/2}\} \ ,(20)$$

we have:

$$\sum_{n_1, n_2, n_3=1}^{\infty} \exp\{-\alpha[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2]^{1/2}\}$$

$$= \frac{1}{8} \sum_{n_1, n_2, n_3=-\infty}^{\infty}{}' \exp\{-\alpha[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2]^{1/2}\}$$

$$- \frac{1}{2} \sum_{n_1, n_2=1}^{\infty} \exp\{-\alpha[(n_1/L_1)^2 + (n_2/L_2)^2]^{1/2}\}$$

$$- \frac{1}{2} \sum_{n_1, n_3=1}^{\infty} \exp\{-\alpha[(n_1/L_1)^2 + (n_3/L_3)^2]^{1/2}\}$$

$$- \frac{1}{2} \sum_{n_1, n_3=1}^{\infty} \exp\{-\alpha[(n_2/L_2)^2 + (n_3/L_3)^2]^{1/2}\}$$

$$- \frac{1}{4} \sum_{n_1=1}^{\infty} \exp[-\alpha n_1/L_1] - \frac{1}{4} \sum_{n_2=1}^{\infty} \exp[-\alpha n_2/L_2] - \frac{1}{4} \sum_{n_3=1}^{\infty} \exp[-\alpha n_3/L_3]$$

By using again the Poisson sumnation formula, we can write

$$\sum_{n_1, n_2, n_3=-\infty}^{\infty} \exp\{-\alpha[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2]^{1/2}\}$$

$$= \sum_{m_1, m_2, m_3=-\infty}^{\infty} \iiint_{-\infty}^{\infty} \exp\{-\alpha[(x/L_1)^2 + (y/L_2)^2 + (z/L_3)^2]^{1/2}$$

$$+ 2\pi i(m_1 x + m_2 y + m_3 z)\}\, dx\, dy\, dz,$$

$$= \iiint_{-\infty}^{\infty} \exp\{-\alpha\left[(x/L_1)^2 + (y/L_2)^2 + (z/L_3)^2\right]^{1/2}\} \, dx\, dy\, dz$$

$$+ \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' \iiint_{-\infty}^{\infty} \exp\{-\alpha\left[(x/L_1)^2 + (y/L_2)^2 + (z/L_3)^2\right]^{1/2}$$

$$+ 2ai \ (m_1x + m_2y + m_3z) \} \, dxdy \, da \quad ,$$

where in the last sum the term $m_1=m_2=m_3=0$ is excluded. Introducing the vectors $\vec{t}$ = (r / ~y/L_2,, a / ~ and) $\vec{\ell} = (m_1L_1, m_2L_2, m_3L_3)$, it follows[3] that

$$\frac{1}{8} \sum_{n_1,n_2,n_3=-\infty}^{\infty} \exp\{-\alpha\left[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2\right]^{1/2}\} =$$

$$\frac{V}{8}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-\alpha t} t^2 \sin\theta \, d\theta \, d\phi \, dt \, +$$

$$+ \frac{V}{8} \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-\alpha t + 2\pi i \ell t \cos\theta} \, t^2 \sin\theta \, d\theta \, d\phi \, dt$$

$$= \frac{\pi V}{2}\int_0^{\infty} t^2 e^{-\alpha t} dt + \pi V \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' \frac{\alpha}{(\alpha^2 + 4\pi^2\ell^2)^2} \ ,$$

with $V = L_1L_2L_3$ and $\ell^2 = (m_1L_1)^2 + (m_2L_2)^2 + (m_3L_3)^2$.

After some calculation, we can write

$$\sum_{n_1,n_2,n_3=1}^{\infty} \exp\{-\alpha\left[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2\right]^{1/2}\} = \frac{\pi V}{2}\int_0^{\infty} t^2 \, e^{-\alpha t} dt \, \text{-}$$

$$- \frac{\pi}{4} (L_1L_2 + L_1L_3 + L_2L_3) \int_0^\infty te^{-\alpha t}\, dt + \frac{1}{4} (L_1 + L_2 + L_3) \int_0^\infty e^{-\alpha t}\, dt$$

$$+ \pi V \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' \frac{\alpha}{(\alpha^2+4\pi^2\ell^2)^2} - \frac{\pi}{4} L_1L_2 \sum_{m_1,m_2=-\infty}^{\infty}{}' \frac{\alpha}{(\alpha^2+4\pi^2\ell_1^2)^{3/2}} -$$

$$- \frac{\pi}{4} L_1L_3 \sum_{m_1,m_3=-\infty}^{\infty}{}' \frac{\alpha}{(\alpha^2+4\pi^2\ell_2^2)^{3/2}} - \frac{\pi}{4} L_2L_3 \sum_{m_2,m_3=-\infty}^{\infty}{}' \frac{\alpha}{(\alpha^2+4\pi^2\ell_3^2)^{3/2}} - \frac{1}{4}$$

$$+ \frac{\alpha}{48} \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right) + 0(\alpha^2), \tag{21}$$

with

$$\ell^2 = (m_1L_1)^2 + (m_2L_2)^2 + (m_3L_3)^2 ,$$

$$\ell_1^2 = (m_1L_1)^2 + (m_2L_2)^2 , \quad \ell_2^2 = (m_1L_1)^2 + (m_3L_3)^2, \quad \ell_3^2 = (m_2L_2)^2 \ (m_3L_3)^2.$$

The first three terms of Eq.(21) give poles in a and are therefore subtracted. Then the regularized expression for Eq.(20) is:

$$\left\{ \sum_{n_1,n_2,n_3=1}^{\infty} \left[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2\right]^{1/2} \right\}^R =$$

$$- \frac{V}{16\pi^3} \sum_{m_1,m_2,m_3=-\infty}^{\infty}{}' (m_1^2L_1^2 + m_2^2L_2^2 + m_3^2L_3^2)^{-2}$$

$$+ \frac{1}{32\pi^2} L_1L_2 \sum_{m_1,m_2=-\infty}^{\infty}{}' (m_1^2L_1^2 + m_2^2L_2^2)^{-3/2} + \frac{1}{32\pi^2} L_1 L_3 \sum_{m_1,m_3=-\infty}^{\infty}{}' (m_1^2L_1^2+m_3^2L_3^2)^{-3/2}$$

$$+ \frac{1}{32\pi^2} L_2L_3 \sum_{m_2,m_3=-\infty}^{\infty}{}' (m_2^2L_2^2 + m_3^2L_3^2)^{-3/2} - \frac{1}{48} \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right) . \tag{22}$$

If we now make the regularization via the zeta function, by considering

$$\left\{ \sum_{n_1,n_2,n_3=1}^{\infty} \left[(n_1/L_1)^2 + (n_2/L_2)^2 + (n_3/L_3)^2\right]^{1/2} \right\}^R =$$

$$\lim_{s\to-\frac{1}{2}} \left\{ \frac{1}{8} A(L_1,L_2,L_3;2s) - \frac{1}{8} A(L_1,L_2;2s) \right.$$

$$\left. - \frac{1}{8} A(L_1,L_3;2s) - \frac{1}{8} A(L_2,L_3;2s) + \frac{1}{4} (L_1^{2s} + L_2^{2s} + L_3^{2s}) \zeta(2s) \right\}, \qquad (23)$$

where $A(L_1,L_2,L_3;2s)$ is the Epstein zeta function, it follows (cf. Appendix B) that the righthand side of Eq.(23) is exactly the righthand side of Eq.(22), which corresponds to substracting from the energy density (in the volume $V = L_1L_2L_3$) the contribution for $V \to \infty$.

Finally, let us remark that in the case of the electromagnetic field in a box $V = L_1L_2L_3$, which was considered by Lukosz[3], both the exponential regularization (where we neglected the pole terms in a), and the analytic continuation via the Epstein zeta functions, give exactly the same results.

6. CONNECTION WITH GREEN FUNCTIONS; DIMENSIONAL REGULARIZATION

Let us for instance consider a scalar field $\phi(x,t)$ which satisfies

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \phi(x,t) = 0 ,$$

where $x = \{x_i\}$, $i = 1,2,\ldots,N$. The eigenfunctions are of the form $\phi_n(x,t) = \phi_n(x)\exp(-i\omega_n t)$, with $n = \{n_i\}$. Now, $\phi_n(x)$, a complete orthonormal set, and ω_n are determined by the boundary conditions. For imaginary times $t = -i\tau$ (τ real), the corresponding Green function writes as:

$$G(x,x';\tau) = \frac{1}{2\pi} \sum_{n=\{n_i\}} \phi_n^*(x')\phi_n(x) \int_{-\infty}^{+\infty} \frac{e^{i\omega\tau}}{\omega^2 + \omega_n^2}\, d\omega \qquad (24)$$

$$G(x,x';\tau) = \frac{1}{2} \sum_{n=\{n_i\}} \frac{\phi_n^*(x')\phi_n(x)\, e^{-\alpha\omega_n}}{\omega_n} \quad , \; \alpha = |\tau| \, .$$

The zero point energy of our system is then defined as

$$\lim_{\alpha\to 0_+} \hbar \frac{\partial^2}{\partial\alpha^2} \int G(x,x'=x;\tau)dx = \lim_{\alpha\to 0_+} \frac{\hbar}{2} \sum_{n=\{n_i\}} \omega_n e^{-\alpha\omega_n} \, , \tag{25}$$

and so the imaginary time τ appears as a regulating parameter. As we know, the righthand side of Eq.(25) has poles for $\alpha=0$.

We would like to have an expression which has no pole at $\alpha=0$. For this purpose, we introduce another parameter, λ, in the Green function given by Eq. (24). We can, for instance, consider

$$G_\lambda(x,x';\tau) = \frac{1}{2\pi} \sum_{n=\{n_i\}} \phi_n^*(x')\phi_n(x) \int_{-\infty}^{\infty} \frac{e^{i\omega^\lambda\tau}}{\omega^2+\omega_n^2}\, d\omega \, ,$$

which is equal, for $0\leq\lambda\leq 1$, to

$$G_\lambda(x,x';\tau) = \frac{1}{2} \sum_{n=\{n_i\}} \phi_n^*(x')\phi_n(x) \frac{e^{-\alpha\omega_n^\lambda}}{\omega_n} \, , \quad \alpha = |\tau| \, .$$

We now extend λ to the region of the complex plane which makes the integral of $G_\lambda(x,x'=x;\tau)$ meaningful. The the zero point energy will be

$$\lim_{\alpha\to 0_+} \hbar \frac{\partial^2}{\partial\alpha^2} \int G_\lambda(x,x'=x;\tau)dx = \lim_{\alpha\to 0_+} \frac{\hbar}{2} \sum_{n=\{n_i\}} \omega_n^{2\lambda-1} e^{-\alpha\omega_n^\lambda} \tag{26}$$

In order to have an idea about the singularity in a in the righthand side of Eq.(26), let us consider the Mellin transform of it:

$$\int_0^\infty \alpha^{p-1} \left(\sum_{n=\{n_i\}} \omega_n^{2\lambda-1} \exp\left[-\omega_n^\lambda\alpha\right] \right) d\alpha = \Gamma(p) \sum_{n=\{n_i\}} \omega_n^{-(p\lambda+1-2\lambda)}$$

(we used the fact that $\omega_n>0$). If, say, $\omega_n=n$ (in some units), then the righthand side of this last equation converges for $\lambda>0$ and $p>2$. We see therefore that, for $\lambda>0$, the righthand side of the sum in Eq.(26) has a singularity at least of second order in a. And, in fact, it is not difficult to show, for the one-dimensional box with $\omega_n=n$, that the sum inthe righthandsideof Eq.(26) isuniformlyconvergent for $\lambda>0$ and $\alpha>0$. It has, for $\lambda=0$, a pole in this variable. If we multiply that sum by λ, we see that it is uniformly convergent for $\alpha>0$ and $\lambda\geq 0$. For Re $\lambda<0$, it is also analytic in this variable, even for $\alpha=0$. Therefore, in this region, we can take a=O without getting poles in a, and we obtain the Riemann zeta function in this region, which afterwards can be continued analytically to the physical region, yielding a finite result which is just the regularized one.

We have not been able, for larger configurations, to continue from a region of the X-plane, which exibits poles (or cuts) in a, to a region without such singularities. Another possibility is to consider the Green-like function

$$G_\lambda(x,x';\tau) = \frac{1}{2\pi} \sum_{n=\{n_i\}} \frac{\phi_n^*(x')\phi_n(x)}{\omega_n^{-\lambda}} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\omega^2+\omega_n^2}\, d\omega$$

$$= \frac{1}{2} \sum_{n=\{n_i\}} \omega_n^{-1+\lambda}\ \phi_n^*(x')\phi_n(x)\ e^{-\alpha\omega_n} \quad , \ \alpha = |\tau| \ . \tag{27}$$

Therefore, the zero point energy is given by

$$\lim_{\alpha\to 0_+} \hbar \frac{\partial^2}{\partial\alpha^2}\int G_\lambda(x,x'=x;\tau)dx = \lim_{\alpha\to 0_+} \frac{\hbar}{2} \sum_{n=\{n_i\}} \omega_n^{1+\lambda}\ e^{-\alpha\omega_n} \ . \tag{27'}$$

For $\alpha>0$, the righthand side is uniformly convergent for any value of λ. In the case of a box of dimensions D, the sum in expression (27') is, for Re $\lambda<-1-D$, analytic in the variable λ even if $a=0_+$. This limit yields functions of the zeta type, which can be continued analytically for Re $\lambda>-1-D$, these functions givíng in many cases, as in those studied in this paper, finite results for the observables.

Let us note that the G_λ function, Eq. (27), satisfies the differential equation

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial x^2}\right)G_\lambda(x,x';\tau) = -\delta_\lambda(x,x')\delta(\tau),$$

where, for real $\phi_n(x)$,

$$\delta_\lambda(x,x') = \sum_{n=\{n_i\}} \frac{\phi_n(x')\phi_n(x)}{\omega_n^{-\lambda}} ,$$

and $\lim_{\lambda\to 0} \delta_\lambda(x,x') = \delta(x-x')$. Here, the regularization procedure amounts to analytically continuing the $\phi_n(x)$ fields, i.e., to consider new fields

$$\phi_n(\lambda,x) = \omega_n^{\lambda/2}\, \phi_n(x) ;$$

the extension to complex $\phi_n(x)$ is not hard to get.

Finally, let us consider the procedure of dimensional regularization[11]. As we have given boundary conditions in space, we will make the analytic continuation in the number of imaginary time dimensions. That is, instead of Eq. (24), we will consider

$$G_\sigma(x,x';\tau_1,\dots,\tau_{2\sigma}) = \frac{1}{(2\pi)^{2\sigma}} \sum_{n=\{n_i\}} \phi_n^*(x')\phi_n(x) \int \frac{d\Omega^{2\sigma}\, e^{i\vec{\Omega}.\vec{\tau}}}{\Omega^2+\omega_n^2} ,$$

where $\Omega^2 = \Omega_1^2 + \dots + \Omega_{2\sigma}^2$, and $\vec{\Omega}.\vec{\tau} = \Omega_1\tau_1 + \dots + \Omega_{2\sigma}\tau_{2\sigma}$. The zero point energy is defined as

$$\lim_{\tau_1,\tau_2,\dots,\tau_{2\sigma}\to 0_+} \hbar \left(\frac{\partial^2}{\partial\tau_1^2}+\dots+\frac{\partial^2}{\partial\tau_{2\sigma}^2}\right) \int G(x,x'=x;\tau_1,\tau_2,\dots,\tau_{2\sigma})\,dx$$

$$= -\frac{\hbar}{(2\pi)^{2\sigma}} \sum_{n=\{n_i\}} \int \frac{d\Omega^{2\sigma}\,\Omega^2}{\Omega^2+\omega_n^2} = -\frac{\pi^\sigma\hbar\sigma\Gamma(-\sigma)}{(2\pi)^{2\sigma}} \sum_{n=\{n_i\}} \omega_n^{2\sigma} ,$$

valid for Re $\sigma < 0$. In particular, for the one-dimensional box, theabove expression is convergent for Re $\sigma < -\frac{1}{2}$. In this way, we obtain the Riemann zeta function which can be continued analytically for Re $\sigma > -\frac{1}{2}$, up to the physical point $\sigma = \frac{1}{2}$.

We thank Profs. C.G. Bollini and R. Aldrovandi for useful discussions.

APPENDIX A – THE RIEMANN ZETA FUNCTION AS FINITE PART OF AN INTEGRAL

The definition of the Riemann zeta function by means of the integral

$$\Gamma(s)\ \zeta(s) = \int_0^\infty \frac{x^{s-1}dx}{e^x - 1} , \tag{A.1}$$

is valid for $\mathrm{Re}\, s > 1$.

In order to make the analytic continuation for $\mathrm{Re}\, s < 1$, we define the lefthand side of Eq.(A.1) as the finite part *à la* Hadamard of the integral in the righthand side. Hadamard's finite part is obtained by subtracting from $(e^x-1)^{-1}$ as many terms from its Laurent expansion (around x = 0) as is necessary to make the integral convergent. For $0 < \mathrm{Re}\, s < 1$, we have[6]

$$\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1} \left(\frac{1}{e^x-1} - \frac{1}{x}\right) dx ,$$

which satisfies the functional equation[6]

$$\Gamma(s)\zeta(s) = \frac{2^{s-1}\ \pi^s}{\cos(\pi s/2)}\ \zeta(1-s) . \tag{A.2}$$

For $-1 < \mathrm{Re}\, s < 0$, we have[6]

$$\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1} \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right) dx$$

which also satisfies the functional equation (A.2). For

$$-1 - 2k < \mathrm{Re}\ s < 1 - 2k , \tag{A.3}$$

with $k \neq 0$, the procedure goes like:

$$\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1}\left[\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} - \frac{1}{12}x + \ldots + \frac{(-1)^k\,|B_{2k}|\,x^{2k-1}}{(2k)!}\right] dx\ .$$

Using the identity[6]

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + 2x\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2+x^2}\ ,$$

it follows then

$$\Gamma(s)\zeta(s) = 2\int_0^\infty x^s\{\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2+x^2} - \frac{|B_2|}{2.(2!)} + \frac{|B_4|x^2}{2.(4!)} + \ldots + \frac{(-1)^k|B_{2k}|x^{2k-2}}{2.(2k)!}\}\,dx\ .$$

Using:

$$\frac{|B_{2k}|}{2.(2k)!} = \sum_{n=1}^{\infty}\frac{1}{(4\pi^2n^2)^k}\ ,$$

it follows that

$$\Gamma(s)\zeta(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^k}{(4\pi^2n^2)^k}\int_0^\infty\frac{x^{s+2k}}{4\pi^2n^2+x^2}\,dx\ ,$$

or

$$\Gamma(s)\zeta(s) = 2\sum_{n=1}^{\infty}(-1)^k\,(2\pi n)^{s-1}\int_0^\infty\frac{y^{s+2k}}{1+y^2}\,dy\ .$$

In the strip given by (A.3), the above integral is convergent, and we obtain

$$\Gamma(s)\zeta(s) = \frac{2^{s-1}\,\pi^s}{\cos(\text{as}/2)}\sum_{n=1}^{m} n^{s-1}\ ,$$

which is exactly the functional equation (A.2).

APPENDIX B – THE EPSTEIN ZETA FUNCTION AND ITS FUNCTIONAL RELATION

Let us consider the following Epstein zeta function[7]:

$$A(a_1,a_2,\ldots,a_D;2s) = \sum_{n_1,n_2,\ldots,n_D=-\infty}^{\infty}{}' (a_1n_1^2+a_2n_2^2+\ldots+a_Dn_D^2)^{-s} ,$$

with the $a_i>0$. It defines an analytic function for Re $s>\frac{D}{2}$ (in the sum, the term $n_1=n_2=\ldots=n_D=0$ has been excluded).

In order to extend this function for Re $s<D/2$, let us derive a functional equation for it. The procedure to be followed is exactly the same as used for the Riemann zeta function[6]. Here, let us consider the gamma function

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1}\, dt$$

and make the change of variables

$$t = \pi(a_1n_1^2 + a_2n_2^2 + \ldots + a_Dn_D^2)x .$$

Then, it follows that

$$\pi^{-s}\,\Gamma(s)\,(a_1n_1^2+a_2n_2^2+\ldots+a_Dn_D^2)^{-s} = \int_0^\infty e^{-\pi(a_1n_1^2+\ldots+a_Dn_D^2)x} \,.\, x^{s-1}\, dx .$$

We have, therefore,

$$\pi^{-s}\,\Gamma(s) \sum_{n_1,n_2,\ldots,n_D=-\infty}^{\infty}{}' (a_1n_1^2+a_2n_2^2+\ldots+a_Dn_D^2)^{-s} =$$

$$\int_0^\infty \sum_{n_1,n_2,\ldots,n_D=-\infty}^{\infty}{}' e^{-\pi(a_1n_1^2+\ldots+a_Dn_D^2)x} \,.\, x^{s-1}\, dx \qquad (B.1)$$

Using the Poisson surnrnation formula, we can write

$$\sum_{n_1,n_2,\ldots,n_D=-\infty}^{\infty} e^{-\pi(a_1n_1^2 + a_2n_2^2 + \ldots + a_Dn_D^2)x} =$$

$$\frac{1}{\sqrt{a_1a_2\ldots a_D}} \frac{1}{x^{D/2}} \sum_{n_1,\ldots n_D=-\infty}^{\infty} e^{-\pi\left(\frac{n_1^2}{a_1} + \ldots + \frac{n_D^2}{a_D}\right)\frac{1}{x}} ,$$

or

$$\sum_{n_1,n_2,\ldots,n_D=-\infty}^{\infty}{}' e^{-\pi(a_1n_1^2 + a_2n_2^2 + \ldots + a_Dn_D^2)x} = -1 + \frac{1}{\sqrt{a_1a_2\ldots a_D}} \frac{1}{x^{D/2}}$$

$$+ \frac{1}{\sqrt{a_1a_2\ldots a_D}} \frac{1}{x^{D/2}} \sum_{n_1,n_2,\ldots n_D=-\infty}^{\infty}{}' e^{-\pi\left(\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2} + \ldots + \frac{n_D^2}{a_D}\right)\frac{1}{x}} . \quad \text{(B.2)}$$

The integration in Eq. **(B.1)** is split into **two** parts: frorn $[0,1]$ and $[1,\infty]$. **In** the first interval, we use **Eq.** (8.2) and we obtain:

$$\pi^{-s}\,\Gamma(s)A(a_1,a_2,\ldots,a_D;2s) = -\int_0^1 x^{s-1}\,dx + \frac{1}{\sqrt{a_1a_2\ldots a_D}}\int_0^1 x^{s-D/2}\,\frac{dx}{x}$$

$$+ \frac{1}{\sqrt{a_1a_2\ldots a_D}} \int_0^1 \sum_{n_1\ldots n_D=-\infty}^{\infty}{}' e^{-\pi\left(\frac{n_1^2}{a_1} + \ldots + \frac{n_D^2}{a_D}\right)\frac{1}{x}} \frac{dx}{x} +$$

$$\int_1^{\infty} \sum_{n_1\ldots n_D=-\infty}^{\infty}{}' e^{-\pi(a_1n_1^2 + \ldots + a_Dn_D^2)x} \times x^s\,\frac{dx}{x} . \quad \text{(B.3)}$$

In the last but one integral of Eq. (B.3), we make the change of variables $y = 1/x$, and obtain:

$$\pi^{-s}\Gamma(s)\ A(a_1,a_2,\dots,a_D;2s) = -\frac{1}{s} + \frac{1}{\sqrt{a_1a_2\dots a_D}}\ \frac{1}{(s-D/2)} +$$

$$+ \frac{1}{\sqrt{a_1a_2\dots a_D}} \int_1^\infty \sum_{n_1\dots n_D=-\infty}^{\infty}{}' e^{-\pi\left(\frac{n_1^2}{a_1}+\frac{n_2^2}{a_2}+\dots+\frac{n_D^2}{a_D}\right)y} \times y^{\frac{D}{2}-s}\frac{dy}{y}$$

$$+ \int_1^\infty \sum_{n_1,\dots n_D=-\infty}^{\infty}{}' e^{-\pi(a_1n_1^2+\dots+a_Dn_D^2)y} \times y^s\frac{dy}{y} .$$

We see that $A(a_1,\dots,a_D;2s)$ has a simple pole for $s = D/2$. Interchanging $s \leftrightarrow \frac{D}{2} - s$, $a_i \leftrightarrow \frac{1}{a_i}$ $(i = 1,2,\dots,D)$, we obtain the following functional relation:

$$\pi^{-(D/2-s)}\ \Gamma(D/2-s)\ A\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_D};\ D-2s\right) \tag{B.4}$$

$$= \sqrt{a_1a_2\dots a_D}\ \pi^{-s}\ \Gamma(s)\ A(a_1,a_2,\dots,a_D;\ 2s) ,$$

which allows us to analytically continue $A(a_1,a_2,\dots,a_D;2s)$ for $\mathrm{Re}\,s < D/2$.

For $a_1 = a_2 = \dots = a_D = 1$, it follows that

$$A(1,1,\dots,1;2s) = \frac{\pi^{2s-D/2}\ \Gamma(D/2-s)}{\Gamma(s)} A(1,1,\dots,1;D-2s) ,$$

which has been obtained by Zucker[12].

REFERENCES

1. H.B.G. Casimir, Proc.K.Ned.Acad.Wetensch.51, 793 (1948); Philips Res. Rep. 6, 162 (1951).

2. M. Fierz, Helv.Phys.Acta 33, 855 (1960).

3. W. Lukosz, Physica 56, 109 (1971).

4. I.M. Guelfand and G.E. Shilov, *Les Distributions* (Dunod, Paris,1962).

5. C.G. Bollini, J.J. Giambiagi and A. Gonzãles Dominguez, Il Nuovo Cimento 31, 550 (1964).

6. E.C. Titchmarch, *The Theory of the Riemann Zeta Function* (Oxford at the Clarendon Press, 1951).

7. P. Epstein, Math. Ann. 56, 615 (1902).

8. After we have finished this paper, it came to our notice a paper by S.W. Hawking, Commum.Math.Phys. 55, 149 (1977), in which the regularization of the Casimir forces by means of generalized zeta functions is also considered. The material contained in Sections 2 and 3, of our paper, including the regularization with the help of zeta functions, was exposed in seminars of ours at the Physics Departments of the *Pontifícia Universidade Católica* (Rio de Janeiro) and the University of São Paulo, in 1973 and 1974 respectively.

9. T.H. Boyer, Phys. Rev. 174, 1764 (1963).

10. B. Davies, J. Math. Phys. 13, 1324 (1972).

11. C.G. Bollini and J.J. Giambiagi, Phys. Lett. *40B*, 566 (1972). See also G. t'Hooft and M. Veltman, Nucl.Phys. *44B*, 189 (1972).

12. I.J. Zucker, J. Phys. A (Math.Gen.) 9, 499 (1976).