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Dirac Particle in a Scalar Coulomb Field

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It is shown that the problem of a Dirac particle in an external potential, which is a superposition of attractive vector and scalar potentials, both of Coulomb form, admits, for bound-states, an exact solution. As particular cases, the bound-state problem for the pure vector Coulomb and scalar Coulomb cases are derived. The degeneracy pattems of the spectra are discussed as well as the role of 0(4,1) as the spectrum generating algebra The magnetic moments of the bound-states, in both scalar and vector cases, are also discussed.

Considerando uma partícula de Dirac em um campo externo produzido pela superposição de um potencial vetorial atrativo e um potencial escalar também atrativo, ambos de forma coulombiana, mostra-se que a equação de Dirac, para estados ligados, admite solução exata Particularizando o presente tratamento, obtemos os casos puramente vetorial e puramente escalar. Discute-se a degenerescência do espectro e o papel de 0(4,1) como a álgebra geradora do espectro. Os momentos magnéticos dos estados ligados para os casos escalar e vetorial são também discutidos.

1. Introduction

Only for a few special cases, the Dirac equation in an external electromagnetic field is known to admit exact solutions[1]. Among them, the most important is that of the Coulomb vector potential, due to its relevance to the spectroscopy of the hydrogen atom.

In this paper, it is shown that the bound-state of a Dirac particle in a Coulomb vector field to which is superimposed a scalar potential of Coulomb form, also admits an exact solution[2]. The degeneracies in the spectrum of the particle in a vector Coulomb field are not removed by the addition of a scalar potential. The two cases of a pure vector Coulomb potential and pure scalar Coulomb potential are special cases of the present treatment. The similarities and main physical differences of these cases are discussed, particularly the characteristics of the energy spectrum, the dege-

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neracy patterns and the magnetic moments of the bound-states in a weak external magnetic field. The role of the Lie algebra 0(4, 1) as the spectrum generating algebra is also discussed.

2. Solutions for Bound-States

The Hamiltonian for a Dirac particle in the presence of an attractive vector potential ($\mathscr{A} = 0, i\mathscr{V}$)

$$e\mathscr{V} = -\frac{\alpha}{r} \tag{1}$$

and an attractive scalar potential

$$\gamma V = -\frac{\gamma^2}{r} \tag{2}$$

reads ($\hbar = c = 1$):

$$H = \alpha \cdot \mathbf{p} - e\mathscr{V} + \beta(m + \gamma V), \tag{3}$$

where a and β are the Dirac matrices which, in the Dirac-Pauli representation, are written as

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the components of a being the Pauli 2 x 2 matrices.

One can immediately see that, in the present case, besides $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}$, the well-known Dirac operator

$$K = \beta(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1) \tag{4}$$

is also an integral of motion, i.e., $[H, K] = 0$. The existence of this integral allows the separation of the angular part of the energy eigenfunctions $u(\mathbf{r})$, in a well-known way[3], through the' Ansatz

$$u(\mathbf{r}) = \begin{pmatrix} g(\mathbf{r}) & \mathscr{Y}^{j_3}_{jl_A} \\ if(\mathbf{r}) & \mathscr{Y}^{j_3}_{jl_B} \end{pmatrix},$$

where the indices l_A and l_B are related to the eigenvalues of K and $\mathbf{J}^2$, which satisfy

$$Ku(\mathbf{r}) = -ku(\mathbf{r}), \qquad \mathbf{J}^2 u(\mathbf{r}) = j(j+1)\,u(\mathbf{r}),$$

by

$$l_A = \begin{cases} j + \frac{1}{2} & \text{for} \quad k = j + \frac{1}{2}, \\ j - \frac{1}{2} & \text{for} \quad k = -(j + \frac{1}{2}), \end{cases}$$

and

$$l_B = \begin{cases} j - \frac{1}{2} & \text{for} \quad k = j + \frac{1}{2}, \\ j + \frac{1}{2} & \text{for} \quad k = -(j + \frac{1}{2}). \end{cases}$$

The radial functions satisfy the system of coupled equations:

$$\frac{df}{dr} + \frac{1-k}{r} f + \left(\frac{\gamma+\alpha}{r} - \alpha_2\right) g = 0,$$

$$\frac{dg}{dr} + \frac{1+k}{r} + \left(\frac{\gamma-\alpha}{r} - \alpha_1\right) f = 0, \tag{5}$$

where

$$\alpha_1 = m + E, \; \alpha_2 = m - E. \tag{6}$$

One can easily see that, for small values of r, both $f(r)$ and $g(r)$ behaves as r^{s-1}, with

$$s \equiv \sqrt{k^2 + \gamma^2 - \alpha^2}. \tag{7}$$

In order to solve Eq. *(5)* we make the following ***Ansatz***[4]***:***

$$f(\sigma) = -\sqrt{\alpha_2}\, e^{-\sigma/2}\, \sigma^{s-1}\,(\varphi - \psi),$$

$$g(\sigma) = \sqrt{\alpha_1}\, e^{-\sigma/2}\, \sigma^{s-1}\,(\varphi + \psi), \tag{8}$$

with $\sigma \equiv 2\sqrt{\alpha_1\alpha_2}\, r$. Substitution of Eq. (8) into Eq. (5) gives the following equations of the Kummer type[5], for $\varphi(r)$ and $\psi(r)$:

$$\sigma\varphi'' + (2s + 1 - \sigma)\,\varphi' - \left(s - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}}\right)\varphi = 0,$$

$$\sigma\psi'' + (2s + 1 - \sigma)\,\psi' - \left(s + 1 - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}}\right)\psi = 0, \tag{9}$$

whose regular solutions at the origin are of the form[5]:

$$\varphi = A \cdot {}_1F_1\left(s - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}},\ 1 + 2s, \sigma\right),$$

$$\psi = -\frac{s - \dfrac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}}}{k - \dfrac{\gamma E + \alpha m}{\sqrt{\alpha_1\alpha_2}}}\, A \cdot {}_1F_1\left(s + 1 - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}},\ 1 + 2s, \sigma\right), \tag{10}$$

in terms of the hypergeometric confluent function ${}_1F_1(a, b, \sigma)$.

Since ${}_1F_1(a, b, \sigma)$ diverges exponentially for large values of σ, one must, in order to have physically admissible solutions for f and g, impose the condition that the ${}_1F_1(a, b, \sigma)$'s in Eq. (10) reduce to polynomials. Thus,

$$s - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}} = -n'; \quad n' = 0, 1, 2, \ldots,$$

$$s + 1 - \frac{\gamma m + \alpha E}{\sqrt{\alpha_1\alpha_2}} = -\bar{n}'; \quad \bar{n}' = 0, 1, 2, \ldots, \tag{11}$$

One must exercise some care when $n' = 0$ (E' $= 1$), since in this case, the series which appears in ψ, does not terminate. However, from the first Eq. (11), and Eq. (7), one sees that $n' = 0$ implies

$$|k| = \frac{\gamma E + \alpha m}{\sqrt{\alpha_1\alpha_2}} \tag{12}$$

In this case, from Eq. (10), one must have

$$-\frac{s-\dfrac{\gamma m+\alpha E}{\sqrt{\alpha_1\alpha_2}}}{k-\dfrac{\gamma E+\alpha m}{\sqrt{\alpha_1\alpha_2}}}A=0, \qquad A\neq 0.$$

This condition follows from Eq. (12) without any indeterminacy only if $k<0$. It follows that, for $n'=0$, one must exclude the case $k>0$, a result which, as we shall see, implies the non-degeneracy of the $n'=0$ levels. From the first of Eqs. (11), it follows that the energy of the bound-states is given by

$$E=m\left(\frac{\sqrt{\gamma^2\alpha^2+[(n'+s)^2+\alpha^2][(n'+s)^2-\gamma^2]}}{(n'+s)^2+\alpha^2}-\frac{\gamma\alpha}{(n'+s)^2+\alpha^2}\right). \tag{13}$$

From Eq. (13), it follows that the degeneracy of the energy spectrum is similar to the well-known case of the Dirac H-atom, namely: the $n'=0$ states are not degenerate since only the $k<0$ values are allowed. For $n'=0$, due to the quadratic dependence on k, there is a double degeneracy of the energy levels, related to the two signs of k. Thus, one sees that the scalar potential does not remove the degeneracy already present in the pure vector case.

3. Pure Coulomb Scalar Potential

As particular cases of our treatment, we have:

i) pure scalar case $\alpha=0$, $\gamma\neq 0$. The energy levels are given by

$$E_{\alpha=0}=m\sqrt{1-\frac{\gamma^2}{(\sqrt{k^2+\gamma^2}+n')^2}}. \tag{14}$$

The corresponding radial functions are

$$f(\sigma) = -C \frac{\sqrt{m-E}}{k - \frac{\gamma E}{\sqrt{m^2-E^2}}} e^{-\sigma/2} \sigma^{s-1} \left[\left(k - \frac{\gamma E}{\sqrt{m^2-E^2}} \right) \cdot {}_1F_1(-n'_\gamma, 1+2s, \sigma) \right.$$

$$\left. - n'_\gamma \cdot {}_1F_1(1-n'_\gamma, 1+2s, \sigma) \right], \tag{14'}$$

$$g(\sigma) = C \frac{\sqrt{m+E}}{k - \frac{\gamma E}{\sqrt{m^2-E^2}}} e^{-\sigma/2} \sigma^{s-1} \left[\left(k - \frac{\gamma E}{\sqrt{m^2-E^2}} \right) \cdot {}_1F_1(-n'_\gamma, 1+2s, \sigma) \right.$$

$$\left. + n'_\gamma \cdot {}_1F_1(1-n'_\gamma, 1+2s, \sigma) \right],$$

where $n'_\gamma = \frac{\gamma m}{\sqrt{m^2-E^2}} - s.$

ii) pure vector case[1] $a \neq 0$, $\gamma = 0$, with **energy** levels given by

$$E_{\gamma=0} = \frac{m}{\sqrt{1 + \frac{\alpha^2}{(\sqrt{k^2-\alpha^2} + n')^2}}}, \tag{15}$$

the corresponding radial functions being

$$f(\sigma) = -C \frac{\sqrt{m-E}}{k - \frac{am}{\sqrt{m^2-E^2}}} e^{-\sigma/2} \sigma^{s-1} \left[\left(k - \frac{\alpha m}{\sqrt{m^2-E^2}} \right) \cdot {}_1F_1(-n'_\alpha, 1+2s, \sigma) \right.$$

$$\left. - n'_\alpha \cdot {}_1F_1(1-n'_\alpha, 1+2s, \sigma) \right] \tag{15'}$$

$$g(\sigma) = C' \frac{\sqrt{m+E}}{k - \frac{\alpha m}{\sqrt{m^2-E^2}}} e^{-\sigma/2} \sigma^{s-1} \left[\left(k - \frac{\alpha m}{\sqrt{m^2-E^2}} \right) \cdot {}_1F_1(-n'_\alpha, 1+2s, \sigma) \right.$$

$$\left. + n'_\alpha \cdot {}_1F_1(1-n'_\alpha, 1+2s, \sigma) \right]$$

with $n'_\alpha = \frac{\alpha E}{\sqrt{m^2-E^2}} - s.$

In Eqs. (14') **and (15')**, C and C are **normalization** constants.

Although Eq. (15) is an exact expression, it is only a meaningful one as far as a $\leq |k|$. On the contrary, Eq. (14) is valid for a scalar Coulomb potential of arbitrarily high strength[6].

Introducing the principal quantum number $n \equiv n' + |k|$, one gets for small values of γ^2, the expression

$$\frac{E_{\alpha=0}}{m} = 1 - \frac{1}{1}\frac{\gamma^2}{n^2}\left(1 - \frac{\gamma^2}{|k|n} + \frac{1}{4}\frac{\gamma^2}{n^2} + \cdots\right) \tag{16}$$

to be compared with the well-known expression for the vector case[3]

$$\frac{E_{\gamma=0}}{m} = 1 - \frac{1}{2}\frac{\alpha^2}{n^2}\left(1 + \frac{\alpha^2}{|k|n} - \frac{3}{4}\frac{\alpha^2}{n^2} + \cdots\right) \tag{17}$$

As pointed out by Lipkin and Tavkhelidze[7], there is, as far as the magnetic moments of the bound-states are concerned, a remarkable difference whether the external potential is of a vector or scalar nature. In the former case, the external magnetic fíeld only causes a shift of the bound-state level, giving practically no contribution to the magnetic moment of the bound-state. For a scalar potential, however, the strong binding provides a remarkable enhancement of the magnetic moment of the bound-state. This enhancement mechanism plays an important rôle in the quark relativistic shell-model of N. N. Bogoliubov *et al.*[2].

It is a simple matter to derive this effect using a perturbative treatment of the iterated Dirac equation in the presence of a weak magnetic externa. field, as done by P. N. Bogoliubov[2]. Following his treatment, it is easily shown that the magnetic moment of the ground-state, for the case of a scalar potential of Coulomb form, is given by

$$\mu = \frac{e\sqrt{1+\gamma^2}}{2m}\left[1 - \frac{1}{3}\left(1 - \frac{1}{\sqrt{1+\gamma^2}}\right)\right], \tag{18}$$

which clearly increases with γ^2

4. A Final Remark

In a recent paper on the non-invariance group for the relativistic hydrogen-atom, Kiefer and Fradkin[8] pointed out that the bound-state solutions provide a Hibert space for a class of unitary irreducible representations (UIR) of the de Sitter O(4, 1) group[9]. The relevant UIR for the problem is that **designated** by $v_{\frac{1}{2},\sigma}$ by Ström[10]. This representation is a particular case of the continuous class $v_{r,\sigma}$, depending on the continuous parameter σ. The fact that $v_{\frac{1}{2},\sigma}$ is the relevant representation may be easily seen by decomposition of the representation space in the chain $O(4, 1) \supset O(4) \supset O(3)$:

$$v_{r,\sigma} = \sum_{k,k'} \oplus \mathscr{H}_{k,k'}, \tag{19}$$

where $k, k' = 0, \frac{1}{2}, 1, \ldots$ and $r = min(k+k')$. The $\mathscr{H}_{k,k'}$ are the representation spaces carrying $(2k+1)(2k'+1)$ – dimensional representations of O(4), whose angular momentum content is given by $|k-k'| \leq j \leq k+k'$. For the particular case $r = \frac{1}{2}$, the O(4) representations appearing in Eq. (19) are given by the points of the following diagram, drawn in the (k, k') plane[10] (Fig. 1).

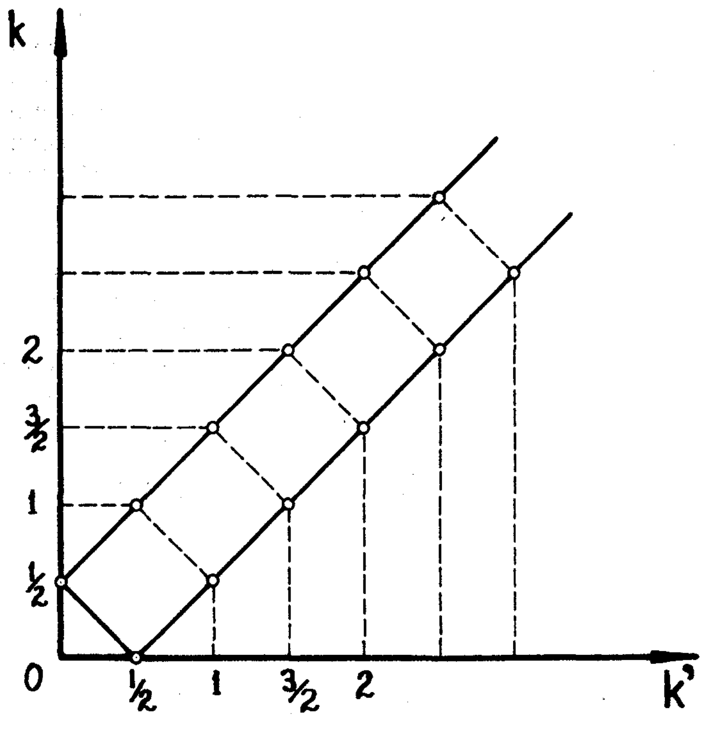

Fig. 1 – The $v_{\frac{1}{2},\sigma}$ representation of O(4, 1) group. The small circles give the allowed values of (k, k') in Eq. (19).

It follows then that

$$\nu_{\frac{1}{2},\sigma} = \mathscr{H}_{0,\frac{1}{2}} \oplus \mathscr{H}_{\frac{1}{2},0} \oplus \mathscr{H}_{\frac{1}{2},1} \oplus \mathscr{H}_{1,\frac{1}{2}} \oplus \mathscr{H}_{1,\frac{3}{2}} \oplus \mathscr{H}_{3,1} \oplus \cdots,$$

whose j content reproduces the well-known pattern of the spectrum. Since in all the three cases given by Eqs. (13), (14) and (15) the structure of the energy spectrum is the same, one concludes that the O(4, 1) group is the non-invariance group, in all these cases, the relevant representations being those of the type $\nu_{\frac{1}{2},\sigma}$. Different values of the parameter σ produce inequivalent representations with the same physical content. Threfore, it is plausible to conjecture that realizations of the Hilbert space of the bound-state solutions for the cases corresponding to Eqs. (13), (14) and (15) are possible, selecting, eventually, differeht values of o. However a constructive proof of the above statement will not be attempted here.

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References and Notes

1. To our knowledge, these are: i) free particle; ii) Coulomb vector potencial (P. A. M. Dirac, Proc. Roy. Soc. A117, 610 (1928)); iii) a constant magnetic field (I. I. Rabi, Zeits. für Physik, 49, 7 (1928)); iv) a constant electric field (F. Sauter, Zeits. für Physik, 69, 742 (1931)); v) the field of a plane wave (D. M. Volkov, Zeits. für Physik, 94, 25 (1935)); vi) plane wave with a constant magnetic field in the direction of propagation (P. J. Redmond, Journ. Math. Phys., 6, 1163 (1965)); vii) four particular configurations of the external electromagnetic field (G. N. Stanciu, Phys. Lett. 23, 232 (1966)).

2. Dirac equation with scalar potentials have been considered by N. N. Bogoliubov and collaborators in the context of a "quark relativistic shell model" in which the scalar potential plays the rôle of an average attractive potential inside the baryon: N. N. Bogoliubov *et al*" JINR D-1968, D-2015, P-2141, Dubna (1965). See also A. Tavkhelidze: *High Energy Phys. and Elementary Particles*, Trieste (1965), pg. 753, edited by IAEA, Vienna 1965; P. N. Bogoliubov, Annales de l'Instiiut Henri Poincaré VIII, 163 (1968).

3. J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley Publ. Co., pg. 127; L. D. Landau and E. M. Lifshitz, *Relativistic Quantum Theory,* Vol. 4, Part. 1, Pergamon Press, N. Y..

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6. For a strong binding, a Klein-Gordon equation with a scalar Coulomb potential shows a different behaviour. As the strength of the potential increases, the low-lying states of the spectrum are gradually removed. This is not the case for the Dirac equation (P. Leal Ferreira, private communication).

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